Math 601 Spring 2015

Homework 3

Due Thursday February 12 at 1pm in Chris Hays mailbox

(1) Let $\sigma\colon E \to B$ be a covering with E locally path connected, and deck group D. Show that D operates properly discontinuously on E. (Do you need that E is locally path connected?).

(2) Let $E \to B$ be a covering and assume that B is a differentiable manifold.
  (a) Show that E is a differentiable manifold such that p is smooth. (I relieve you of the duty to show that E is also second countable, which is more involved).
  (b) Show that the deck group of E acts smoothly on E.
  (c) Show that any cover of a Lie group is a Lie group.

(3) Let $p\colon E \to G$ be a covering of topological groups (or Lie groups) with E path connected. This includes by definition that p is a group homomorphism as well.
  (a) Show that $\ker p$ is a discrete subgroup of $Z(E)$, the center of E.
  (b) Show that p is a normal (regular) cover. What is the deck group?
  (c) Show conversely that if G is a topological group and $\Gamma \subset Z(G)$ a discrete subgroup, then the projection $G \to G/\Gamma$ is a covering.

(4) Let $\mathbb{H}$ be the algebra of quaternions with norm $|q|^2 = a^2 + b^2 + c^2 + d^2$ for $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$.
  (a) Show that $\mathbb{S}^3 = \{q \in \mathbb{H} \mid |q| = 1\}$ is a Lie group (sometimes also called $\mathrm{Sp}(1)$).
  (b) Show that $\sigma\colon \mathbb{S}^3 \to \mathrm{SO}(3)$ is a 2-fold cover, where $\sigma(q)(v) = qvq^{-1}$ for $v \in \operatorname{Im}\mathbb{H}$ (first make sense out of this definition).
  (c) Show that $\sigma\colon \mathbb{S}^3 \times \mathbb{S}^3 \to \mathrm{SO}(4)$ is a 2-fold cover, where $\sigma((q_1, q_2))(v) = q_1 v q_2^{-1}$ for $v \in \mathbb{H}$.

(5) Write the Klein bottle as the union of 2 triangles (with appropriate identifications of the sides). Compute its simplicial homology, both with with $\mathbb{Z}$ coefficients as well as $\mathbb{Z}_p$ coefficients for p a prime. Notice that this is strictly speaking not a triangulation (why not?) but the formulas for computing homology is the same.

(6) (Extra Credit) Show that $\mathrm{SO}(4)$ and $\mathbb{S}^3 \times \mathrm{SO}(3)$ are Lie groups with the same fundamental group. Furthermore show that they are diffeomorphic as manifolds but not isomorphic as Lie groups (or even as groups).