Math 601 Spring 2015

Homework 8

Due Thursday April 2 at 1pm in Chris Hays mailbox

(1) Show that $H^1(X, G)$ is isomorphic to $\mathrm{Hom}(H_1(X, Z), G)$ without using the universal coefficient theorem.

(2) (a) Let X be the disjoint union of spaces X_α. Compute the cohomology ring $H^*(X, R)$ in terms of the cohomology rings $H^*(X_\alpha, R)$.
(b) Let X be the wedge product of spaces X_α (at a point which is a deformation retract of a neighborhood). Compute the cohomology ring $H^*(X, R)$ in terms of the cohomology rings $H^*(X_\alpha, R)$. (See Hatcher Example 3.13).

(3) Determine the ring structure in $H^*(X, \mathbb{Z}_2)$ for the Klein bottle X. For this, write it as a Δ complex with two simplices as we did for homology. What about the ring structure with $\mathbb{Z}$ coefficients?

(4) Assume that X is the union of 2 contractible open subsets. Show that all cup products of 2 classes of positive degree are equal to 0.

(5) Recall the definition of the Hopf invariant for $f\colon \mathbb{S}^{2n-1} \to \mathbb{S}^n$ with $n > 1$: Take the natural generators in cohomology (as discussed in class) $[\alpha], [\beta]$ in $H^{2n-1}(\mathbb{S}^{2n-1}, \mathbb{Z})$ and $H^n(\mathbb{S}^n, \mathbb{Z})$ and write $\beta \cup \beta = \delta u$ and $f^*(\beta) = \delta v$. Then $\gamma = f^*(u) - v \cup f^*(\beta)$ is co-closed and hence $[\gamma] = H(f)\alpha$ for some integer $H(f)$ called the Hopf invariant. Verify again that $\delta\gamma = 0$, and show that $H(f)$ is well defined, i.e. does not depend on the choice of α, β, u, and v.

(6) (Extra Credit) Show that the above definition of the Hopf invariant is equivalent to the second one I gave in class: Form a CW complex X where f is the attaching map from a 2n cell to its n-skeleton $\mathbb{S}^n$. Chose generators $[\alpha], [\beta]$ in $H^n(X, \mathbb{Z})$ and $H^{2n}(X, \mathbb{Z})$ and show that $[\alpha]^2 = H(f)[\beta]$.