Math 601 Spring 2015

Midterm Solutions

(1) Show that $X$ is simply connected iff for any 2 points $p, q \in X$, any two paths from $p$ to $q$ are homotopic to each other.

**Solution:** $\Rightarrow$ Let $\alpha, \beta$ be 2 paths from $p$ to $q$. Then $\alpha \ast \beta \simeq 1_p$ and hence $\alpha \simeq \alpha \ast (\beta \ast \beta) \simeq (\alpha \ast \beta) \ast \beta \simeq 1_p \ast \beta \simeq \beta$.

$\Leftarrow$ If $\alpha \in \pi_1(X, p)$, then $\alpha \simeq 1_p$ since they have the same endpoints.

(2) Compute the fundamental group and homology groups of $\mathbb{R}^n$ with a line taken out. You can assume $n \geq 3$, but you need to present a complete proof of your claim.

**Solution:** We can assume the line is $x_n = 0$. Then $\mathbb{R}^n - L$ deformation retracts to $\mathbb{R}^{n-1} - \{0\}$ via the homotopy $F(x, t) = (x_1, \ldots, x_{n-1}, tx_n)$, which deformation retracts to $S^{n-2}$ via the homotopy $G(x, t) = tx + (1-t)x/||x||$. Thus $\pi_1 = \mathbb{Z}$ if $n = 3$ and $0$ other wise. Similarly, $H_k = \mathbb{Z}$ for $k = 0, n-2$ and $0$ otherwise.

(3) Show that every continuous map $f : \mathbb{S}^2 \to \mathbb{S}^1$ is homotopic to a constant map.

**Solution:** Since $\pi_1(\mathbb{S}^2) = 0$, there exists a lift $\tilde{f} : \mathbb{S}^2 \to \mathbb{R}$ under the cover $p : \mathbb{R} \to \mathbb{S}^1$, i.e. $p \circ \tilde{f} = f$. Since $\mathbb{R}$ is contractible, there is a null homotopy $F : \mathbb{S}^2 \times [0, 1] \to \mathbb{R}$, and then $p \circ F$ is a null homotopy of $f$.

(4) Let $p_i : Y_i \to X_i$, $i=1,2$ be covering spaces with $Y_i$ simply connected. Show that if $X_1$ and $X_2$ are homotopy equivalent, then so are $Y_1$ and $Y_2$.

**Solution:** Let $f : X_1 \to X_2$ and $g : X_2 \to X_1$ be such that $f \circ g \simeq \text{Id}$ and $g \circ f \simeq \text{Id}$. Since $\pi_1(Y_i) = 0$, there exists a lift $\tilde{f}$ of the composition $f \circ p_1$ under the cover $p_2$ and a lift of $\tilde{g}$ of the composition $g \circ p_2$ under the cover $p_1$. Then $\tilde{g} \circ \tilde{f}$ is a lift of $g \circ f \circ p_1$ under the cover $p_1$. The homotopy lifting property then implies that $\tilde{g} \circ \tilde{f} \simeq \text{Id}$, and similarly $\tilde{f} \circ \tilde{g} \simeq \text{Id}$.

(5) Let $X$ be the $\Delta$ complex obtained by starting with two simplices $[v_0, v_1, v_2]$ and $[w_0, w_1, w_2]$ and identifying corresponding vertices, i.e. $v_i \sim w_i$. Compute its homology.

**Solution:** We have the chain complex $0 \to C_2 \simeq \mathbb{Z}^2 \to C_1 \simeq \mathbb{Z}^6 \to C_0 \simeq \mathbb{Z}^3$. Let $\alpha = [v_0, v_1, v_2], \beta = [w_0, w_1, w_2]$ be a basis of $C_2$, and $a = [v_1, v_2], b = [v_0, v_2], c = [v_0, v_1], \bar{a} = [w_1, w_2], \bar{b} = [w_0, w_2], \bar{c} = [w_0, w_1]$ a basis of $C_1$, and $v_0 = w_0, v_1 = w_1, v_2 = w_2$ a basis of $C_0$. Since $\partial_2(\alpha) = a - b + c$ and $\partial_2(\beta) = \bar{a} - \bar{b} + \bar{c}$, the boundary map $\partial_2$ is clearly injective, and hence $H_2(X) = 0$.

$\partial_1(a) = v_2 - v_1 = \delta_1(\bar{a})$ and similarly for $b, c$. Thus $v_0, v_1, v_2$ are homologous, and $H_0(X) = \mathbb{Z}$. The kernel $\ker \partial_1$ has a $\mathbb{Z}$ basis $\{a - \bar{a}, b - \bar{b}, a - b + c, \bar{a} - \bar{b} + \bar{c}\}$ and hence $H_2(X) = \mathbb{Z} \oplus \mathbb{Z}$ with basis $[a - \bar{a}], [b - \bar{b}]$. 
(6) Take the torus $S^1 \times S^1$ and identify points in the circle $S^1 \times \{x_0\}$ that differ by a $2\pi/m$ rotation and identify points in the circle $\{x_0\} \times S^1$ that differ by a $2\pi/n$ rotation. Make it into a CW complex and compute its homology.

**Solution:** The CW complex has one 2-cell $e^2$ and two 1-cells $e_1^1 = \{(x_0, e^{it}) | 0 \leq t \leq 2\pi/m\}$ and $e_2^1 = \{(e^{it}, x_0) | 0 \leq t \leq 2\pi/n\}$ and one 0-cell. We have boundary maps $\partial(e^2) = n e_1^1 + m e_2^1 - n e_1^1 - m e_2^1 = 0$ and $\partial(e_1^1) = \partial(e_2^1) = 0$ and hence $H_0(X) = H_2(X) = \mathbb{Z}$ and $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}.$

(7) Let $A$ be a 3x3 matrix with real positive entries. Show that $A$ has a real positive eigenvalue.

**Hint:** You may want to consider the set

$$A := \{v = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum x_i^2 = 1, \ x_i \geq 0\}.$$ 

**Solution:** Let $f : A \rightarrow A$ be defined by $f(v) = Av/\|Av\|$ which is well defined since $A$ has positive entries. By the Brower fixed point theorem, there exists a $v \in A$ with $f(v) = v$ and hence $Av = \|Av\|v.$

(8) (Extra credit if you have time left) Consider a 3 simplex $[v_0, v_1, v_2, v_3]$ with opposite edges identified (orientation preserving). What is the homology of this space.

**Solution:** $C_3$ is spanned by $a = [v_0, v_1, v_2, v_3]$ and $C_2$ by

$$a = [v_1, v_2, v_3], b = [v_0, v_2, v_3], c = [v_0, v_1, v_3], d = [v_0, v_1, v_2]$$

and $C_1$ by

$$u = [v_0, v_1] = [v_2, v_3], v = [v_0, v_2] = [v_1, v_3], w = [v_0, v_3] = [v_1, v_2].$$

All vertices are identified to one point $[v_0]$ and hence $H_0(X) = \mathbb{Z}.$ We have $\partial_3(a) = a - b + c - d$ and

$$\partial_2(a) = u - v + w = \partial_2(d), \quad \partial_2(b) = u + v - w = \partial_2(c), \quad \partial_1 = 0.$$ 

Thus $H_3(X) = 0$ and

$$H_2(X) = \ker(\partial_2)/\text{Im}(\partial_3) = \langle a - d, b - c \rangle / \langle a - d = b - c \rangle \cong \mathbb{Z}.$$ 

Finally,

$$H_1(X) = \ker(\partial_1)/\text{Im}(\partial_2) = \langle u, v, w \rangle / \langle u - v + w = 0, u + v - w = 0 \rangle$$

$$= \langle u, v - w, w \rangle / \langle 2u = 0, 2(v - w) = 0 \rangle = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$ 

**Correction** (thanks to Zhen Zeng! Exercise: Find my mistake.....)

$$H_1(X) = \ker(\partial_1)/\text{Im}(\partial_2) = \langle u, v, w \rangle / \langle u - v + w = 0, u + v - w = 0 \rangle$$

$$= \langle u, u - v + w, w \rangle / \langle 2u = 0, u - v + w = 0 \rangle = \mathbb{Z} \oplus \mathbb{Z}.$$