(1) Show that \( \text{det} : \text{GL}(n, \mathbb{R}) \to (\mathbb{R}\setminus\{0\}, \cdot) \) is a Lie group homomorphism with \( d\text{det} = \text{tr} \).

**Solution:** Since \( \text{det}(AB) = \text{det}(A)\text{det}(B) \), it is a homeomorphism and is smooth since it is polynomial in terms of the coefficients of \( A \). Since \( \text{det}(e^A) = e^{\text{tr}A} \) (Show this first for diagonalizable matrices and then for all by denseness). Replacing \( A \) by \( tA \) and differentiating gives \( d(\text{det})_{\text{Id}}(A) = \text{tr} A \).

(2) Classify all two dimensional Lie groups.

**Solution:**\( [g, g] \) has dimension 0 or 1. In the first case, \( g \) is abelian, and in the second case there exists a basis \( u, v \) with \( [u, v] = v \). If \( g \) is abelian, then \( G \) is abelian and hence \( G = \mathbb{R}^2, \mathbb{R} \times S^1 \) or \( T^2 \). In the second case, the simply connected Lie group with Lie algebra \( g \) is the Lie group in Problem 9 (for \( n = 1 \)). One easily checks it has trivial center, and hence in this case there is only one Lie group with Lie algebra \( g \).

(3) Let \( H \) be a connected Lie subgroup of a connected Lie group \( G \). Show that the Lie algebra of the normalizer \( N_G(H) = \{ g \in G \mid gHg^{-1} \subset H \} \) is the normalizer \( \{ X \in g \mid [X, h] \subset h \} \).

**Solution:** \( X \) lies in the Lie algebra of \( N_G(H) \) iff \( \exp(tX) \in N_G(H) \) for all \( t \), i.e. \( C_{\exp(tX)}(H) \subset H \) (Notice this is equivalent to equality since both groups are connected). This holds iff \( C_{\exp(tX)}(\exp(sZ)) \subset H \) for all \( s, t \) and \( Z \in h \) since \( H \) is generated by a neighborhood of \( e \). But \( C_{\exp(tX)}(\exp(sZ)) = \exp(s \text{Ad}(\exp(tX))Z) \) and this lies in \( H \) for all \( s \) iff \( \text{Ad}(\exp(tX))Z \in h \). Finally \( \text{Ad}(\exp(tX))Z = e^{t\text{ad}X}(Z) \) implies that this holds iff \( [X, Z] \in h \) for all \( Z \), i.e. \( [X, h] \subset h \).

(4) Determine the image of \( \exp : \mathfrak{sl}(2, \mathbb{R}) \to \text{SL}(2, \mathbb{R}) \). Is it onto or one to one?

**Solution:** Since \( A e^B A^{-1} = e^{ABA^{-1}} \), it follows that \( \exp \) takes conjugacy classes to conjugacy classes. On the Lie algebra level we have the 3 conjugacy classes:

\[
\left( \begin{array}{cc} a & 0 \\ 0 & -a \end{array} \right), \quad \left( \begin{array}{cc} 0 & a \\ 0 & 0 \end{array} \right), \quad \left( \begin{array}{cc} 0 & a \\ -a & 0 \end{array} \right)
\]

with \( a \in \mathbb{R} \)

with exponential image

\[
\left( \begin{array}{cc} e^a & 0 \\ 0 & e^{-a} \end{array} \right), \quad \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right), \quad \left( \begin{array}{cc} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{array} \right)
\]

On the other hand, on the Lie group level we have the conjugacy classes

\[
\left( \begin{array}{cc} \lambda & 0 \\ 0 & 1/\lambda \end{array} \right), \quad \left( \begin{array}{cc} \epsilon & a \\ 0 & \epsilon \end{array} \right), \quad \left( \begin{array}{cc} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{array} \right), \quad \lambda \neq 0, \epsilon = \pm 1
\]

Thus the only matrices that are not in the image of \( \exp \) are

\[
\left( \begin{array}{cc} \lambda & 0 \\ 0 & 1/\lambda \end{array} \right), \quad \left( \begin{array}{cc} -1 & a \\ 0 & -1 \end{array} \right)
\]

with \( \lambda < 0 \).
and their conjugates. The first conjugacy class can be characterized by \( \text{tr} A < -2 \), whereas \( \text{tr} A = -2 \) contains 2 conjugacy classes not in the image (besides \( \text{Id} \in \text{Im exp} \)). The rotations show that \( \exp \) is not injective.

(5) Find an automorphism of \( SO(2n) \) which is not inner.

**Solution:** Let \( A \in O(n) \) but \( A \notin SO(n) \), and \( \phi = C_A \). \( \phi \) is an automorphism since conjugation is, and it preserves the identity component. Thus it lies in \( \text{Aut}(SO(n)) \). If \( C_A = C_B \) with \( B \in SO(n) \), then \( C_{AB^{-1}} = \text{Id} \), i.e. \( AB^{-1} \in Z(O(n)) = \{ \pm \text{Id} \} \). Hence \( B = \pm A \) and \( \det B = -1 \) implies \( B = A \). Notice that this does not work for \( SO(2n+1) \), in fact all automorphisms of \( SO(2n+1) \) are inner.

(6) Show that in the polar decomposition of \( A \in O(p, q) \), \( p, q \geq 1 \), the orthogonal matrix \( R \) lies in \( O(p) \times O(q) \).

**Solution:** Following the proof of Proposition 2.26 for the polar decomposition of \( \text{Sp}(n, \mathbb{R}) \), and replacing \( J \) by \( I_{p,q} \), we see that \( A = RS \) with \( A \in O(p, q) \) and \( R \in O(p+q) \) implies that \( R^T I_{p,q} R = I_{p,q} \). Since \( R^T = R^{-1} \) this means that \( R \) commutes with \( I_{p,q} \) and easily implies that \( R \) must have block form \( R = \text{diag}(L, M) \). But \( R \in O(p+q) \) shows that \( L \in O(p) \), \( M \in O(q) \).

(7) Show that a Lie algebra is semisimple if and only if it has no abelian ideal. This should be proved directly from the definition that \( g \) is semisimple if it has no solvable ideals (i.e., without quoting other theorems).

**Solution:** \( g \) is semisimple if it has no solvable ideals. So clearly, if it has an abelian ideal, it is not semisimple. For the converse, assume that \( \mathfrak{a} \) a \( k \)-step solvable ideal of \( g \). Then \( \mathfrak{a}_{k-1} \) is an abelian ideal of \( \mathfrak{a} \). We need to show it is an ideal in \( g \) as well. But this follows by induction on the descending series \( \mathfrak{a}_i \), \( i = 0, \ldots , k-1 \) using the Jacobi identity: \( [\mathfrak{g}, \mathfrak{a}_{i+1}] = [\mathfrak{g}, [\mathfrak{a}_i, \mathfrak{a}_i]] \subset [\mathfrak{a}_i, [\mathfrak{g}, \mathfrak{a}_i]] \subset [\mathfrak{a}_i, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1} \).

(8) If \( \mathfrak{a} \) is an abelian ideal of \( g \), show that \( \mathfrak{a} \subset \ker B \), where \( B \) is the Killing form of \( g \).

**Solution:** This is the second part of the proof of Theorem 3.19.

(9) Let \( G = \mathbb{R}^n \times \mathbb{R}^+ \) with multiplication \( (x, t) \cdot (y, s) = (x + ty, ts) \).

(a) Show that \( G \) is a Lie group.

(b) Compute the left invariant vector fields \( X_i \), \( T \) whose value at the identity is the standard basis of \( \mathbb{R}^n \) respectively \( \mathbb{R} \).

(c) Compute the Lie algebra of \( G \). Is \( G \) nilpotent, solvable or semisimple?

(d) Compute the Killing form of \( G \).

**Solution:** (a) One can either prove that \( G \) is a Lie group directly, or observe that it agrees with the multiplication in the matrix group

\[
\begin{pmatrix}
t & x \\
0 & \text{Id}_n
\end{pmatrix}
\]

where \( x \) is regarded as a row vector.
(b) $L_{x,t}$ is linear and hence has derivative $t \text{Id}$. This implies that $X_i = t e_i$ and $T = t \frac{d}{dt}$.

(c) Using the definition of Lie brackets in local coordinates, one shows that the only non-vanishing Lie brackets are $[T, X_i] = X_i$. This implies that $\mathfrak{g}$ is solvable (and hence not semisimple), but not nilpotent.

(d) Since $X_i$ span an abelian ideal $B(X_i, X_j) = B(X_i, T) = 0$ by Problem. But $\text{ad}^2 (X_i) = X_i$ and hence $B(T, T) = n$.

(10) For the geometers among you, consider the Lie group from Problem 9 and put a left invariant metric on it such that $X_i, T$ is an orthonormal basis at every point. Show that this metric is isometric to hyperbolic space $\mathbb{H}^{n+1}$ and that $G$ acts by isometries.

Solution: In the coordinates $x_i, t$ with coordinates vector fields $e_i, \frac{d}{dt}$, part (b) implies that $||e_i|| = ||\frac{d}{dt}|| = 1/t$ and $\langle X_i, \frac{d}{dt} \rangle = 0$. Thus the metric is given by

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2 + dt^2}{t^2}$$

which is precisely the hyperbolic metric in the upper half space model.