ASSIGNMENT # 1
MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let \((\mathbb{R}^n, g_0)\) be the (canonical) euclidean metric \(g_0(X,Y) = \sum x_iy_i\). Show that every isometry of this Riemannian manifold is a composition of a rotation, i.e. an element in O\((n)\), and a translation.

(2) Let \((S^n, g_0)\) be the metric on \(S^n\) induced from its usual embedding into \(\mathbb{R}^{n+1}\): \(x_0^2 + \cdots + x_n^2 = 1\).
   (a) What is the isometry group of \(g_0\)?
   (b) Show that if \(G\) is a group of isometries that acts freely on \((S^{2n}, g_0)\), then \(G = \pm \text{Id}\).

(3) Assume that \((M, g)\) is a Riemannian manifold.
   (a) If \(\pi: \tilde{M} \rightarrow M\) is a covering, show that \(\tilde{M}\) admits a Riemannian metric such that \(\pi\) is a local isometry. If the cover is regular, show that the deck group acts by isometries.
   (b) Assume that \(H\) acts on \(M\) properly discontinuously and freely, i.e. \(\pi: M \rightarrow M/H\) is a covering. Assume furthermore that \(H\) acts by isometries. Show that \(M/H\) inherits a Riemannian metric and that \(\pi\) is a local isometry.

(4) Define the ”canonical” metric on \(\mathbb{R}P^n\) and determine its isometry group.

(5) Show that a group \(G\) of isometries acting properly discontinuously and freely on \((\mathbb{R}^2, g_0)\) is one of the following:
   (a) \(G \simeq \mathbb{Z}^2\) and the action of \(G\) on \(\mathbb{R}^2\) is given by \((m,n) \ast v = v + me_1 + ne_2\) where \((m,n) \in \mathbb{Z}^2\) and \(e_1, e_2\) are linearly independent vectors in \(\mathbb{R}^2\). In this case \(\mathbb{R}^2/G\) is a 2-torus.
   (b) \(G \simeq \mathbb{Z}\) with action \(m \ast v = v + me_1\) where \(m \in \mathbb{Z}\) and \(0 \neq e_1 \in \mathbb{R}^2\). In this case \(\mathbb{R}^2/G\) is a cylinder.
   (c) \(G \simeq \mathbb{Z}\) with action \(m \ast (x,y) = (x+ma,-y)\) where \(m \in \mathbb{Z}\), \((x,y) \in \mathbb{R}^2\) and \(0 \neq a \in \mathbb{R}\). In this case \(\mathbb{R}^2/G\) is a Moebius band.
   (d) \(G\) is generated by \(\alpha, \beta\) with relation \(\beta \alpha \beta^{-1} = \alpha^{-1}\) and action \(\alpha(x,y) = (x+a,y)\) and \(\beta(x,y) = (-x,y+b)\) with \(a, b \in \mathbb{R} \setminus \{0\}\). In this case \(\mathbb{R}^2/G\) is a Klein bottle.
   In case (c) and (d) the precise statement is that there exists an orthonormal basis, in whose coordinates the action takes on the given form.