ASSIGNMENT # 1

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let $(\mathbb{R}^n, g_0)$ be the (canonical) euclidean metric $g_0(X,Y) = \sum x_i y_i$. Show that every isometry of this Riemannian manifold is a composition of a rotation, i.e. an element in $\mathrm{O}(n)$, and a translation.

(2) Let $(\mathbb{S}^n, g_0)$ be the metric on $\mathbb{S}^n$ induced from its usual embedding into $\mathbb{R}^{n+1}$: $x_0^2 + \cdots + x_n^2 = 1$.
   (a) What is the isometry group of g_0?
   (b) Show that if G is a group of isometries that acts freely on $(\mathbb{S}^{2n}, g_0)$, then $G = \pm \mathrm{Id}$.

(3) Assume that (M, g) is a Riemannian manifold.
   (a) If $\pi\colon \tilde{M} \to M$ is a covering, show that $\tilde{M}$ admits a Riemannian metric such that π is a local isometry. If the cover is regular, show that the deck group acts by isometries.
   (b) Assume that H acts on M properly discontinously and freely, i.e. $\pi\colon M \to M/H$ is a covering. Assume furthermore that H acts by isometries. Show that M/H inherits a Riemannian metric and that π is a local isometry.

(4) Define the "canonical" metric on $\mathbb{RP}^n$ and determine its isometry group.

(5) Show that a group G of isometries acting properly discontinously and freely on $(\mathbb{R}^2, g_0)$ is one of the following:
   (a) $G \simeq \mathbb{Z}^2$ and the action of G on $\mathbb{R}^2$ is given by $(m,n) \star v = v + me_1 + ne_2$ where $(m,n) \in \mathbb{Z}^2$ and e_1, e_2 are linearly independent vectors in $\mathbb{R}^2$. In this case $\mathbb{R}^2/G$ is a 2-torus.
   (b) $G \simeq \mathbb{Z}$ with action $m \star v = v + me_1$ where $m \in \mathbb{Z}$ and $0 \neq e_1 \in \mathbb{R}^2$. In this case $\mathbb{R}^2/G$ is a cylinder.
   (c) $G \simeq \mathbb{Z}$ with action $m \star (x,y) = (x + ma, -y)$ where $m \in \mathbb{Z}, (x,y) \in \mathbb{R}^2$ and $0 \neq a \in \mathbb{R}$. In this case $\mathbb{R}^2/G$ is a Moebius band.
   (d) G is generated by α, β with relation $\beta\alpha\beta^{-1} = \alpha^{-1}$ and action $\alpha(x,y) = (x+a, y)$ and $\beta(x,y) = (-x, y+b)$ with $a, b \in \mathbb{R}\backslash\{0\}$. In this case $\mathbb{R}^2/G$ is a Klein bottle.

   In case (c) and (d) the precise statement is that there exists an orthonormal basis, in whose coordinates the action takes on the given form.