ASSIGNMENT # 4
MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let \( M \) be a surface in \( \mathbb{R}^3 \).
   (a) If a surface in \( \mathbb{R}^3 \) is given by a parametrization \((x_1, x_2) \rightarrow f(x_1, x_2) \in \mathbb{R}^3\), show that the Gauss curvature is equal to \( \det(h_{ij}) = \det(g_{ij}) \), where \( g_{ij} = \langle f_{x_i}, f_{x_j} \rangle \) and \( h_{ij} = \langle f_{x_i}, x_j, N \rangle \) with \( N = (f_{x_1} \times f_{x_2})/||f_{x_1} \times f_{x_2}||^2 \) a normal to the surface.
   (b) Compute the Gauss curvature of the hyperboloid \( f(t, s) = (\cos s + t \sin s, \sin s - t \cos s, t) \) and show it is generated by 2 families of straight lines.

(2) (a) Show that the curvature of the metric \( dr^2 + f^2(r, \theta)d\theta^2 \) is given by \( -f_{rr}/f \).
   (b) On a surface we can introduce polar coordinates \((r, \theta)\) at \( p \) by
   \[
   (r, \theta) \rightarrow \exp_p(r \cos \theta e_1 + r \sin \theta e_2)
   \]
   where \( e_1, e_2 \) is an orthonormal basis of \( T_pM \). Show that the metric in polar coordinates takes on the form \( dr^2 + f^2(r, \theta)d\theta^2 \).
   (c) Show that locally every function on a surface is the curvature of some metric.

(3) (a) Show that the curvature of the metric \( e^{2u}(dx^2 + dy^2) \) is given by \(-\Delta u\) where \( \Delta u = u_{xx} + u_{yy} \). These coordinates are called isothermal coordinates and one can show that they always exist locally.
   (b) Show that the curvature of hyperbolic space \( \frac{dx^2 + dy^2}{y^2} \) is \(-1\).

(4) (a) Show that the curvature of the graph of \( f \), i.e. set set \((x, y, f(x, y))\) is given by \( \det(\text{Hess}(f))/(1 + f_x^2 + f_y^2)^2 \).
   (b) Show that if the Gauss curvature of \( M \) is positive at \( p \), then locally \( M \) lies on one side of the tangent plane \( T_pM \).

(5) (a) Show that the catenoid \( x(u, v) = (\cosh v \cos u, \cosh v \sin u, v) \) and the helicoid \( y(u, v) = (\sinh v \cos u, \sinh v \sin u, u) \) have the same Gauss curvature.
   (b) Show that \((\cos t)x + (\sin t)y\) is a one parameter family of minimal surfaces (i.e. \( \text{tr}(S) = 0 \)) and that they are all isometric to each other.