ASSIGNMENT # 4

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let M be a surface in $\mathbb{R}^3$.
   (a) If a surface in $\mathbb{R}^3$ is given by a parametrization $(x_1, x_2) \to f(x_1, x_2) \in \mathbb{R}^3$, show that the Gauss curvature is equal to $\det(h_{ij})/\det(g_{ij})$, where $g_{ij} = \langle f_{x_i}, f_{x_j} \rangle$ and $h_{ij} = \langle f_{x_i,x_j}, N \rangle$ with $N = (f_{x_1} \times f_{x_2})/||f_{x_1} \times f_{x_2}||^2$ a normal to the surface.
   (b) Compute the Gauss curvature of the hyperboloid $f(t, s) = (\cos s + t \sin s, \sin s - t \cos s, t)$ and show it is generated by 2 families of straight lines.

(2) (a) Show that the curvature of the metric $dr^2 + f^2(r, \theta)d\theta^2$ is given by $-f_{rr}/f$.
   (b) On a surface we can introduce polar coordinates (r, θ) at p by

$$(r, \theta) \to \exp_p(r \cos\theta \, e_1 + r \sin\theta \, e_2)$$

   where e_1, e_2 is an orthonormal basis of T_pM. Show that the metric in polar coordinates takes on the form $dr^2 + f^2(r, \theta)d\theta^2$.
   (c) Show that locally every function on a surface is the curvature of some metric.

(3) (a) Show that the curvature of the metric $e^{2u}(dx^2 + dy^2)$ is given by $-\triangle u$ where $\triangle u = u_{xx} + u_{yy}$. These coordinates are called isothermal coordinates and one can show that they always exist locally.
   (b) Show that the curvature of hyperbolic space $\frac{dx^2+dy^2}{y^2}$ is -1.

(4) (a) Show that the curvature of the graph of f, i.e. set set $\{(x, y, f(x, y))\}$ is given by $\det(Hess(f))/(1 + f_x^2 + f_y^2)^2$.
   (b) Show that if the Gauss curvature of M is positive at p, then locally M lies on one side of the tangent plane T_pM.

(5) (a) Show that the catenoid $x(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$ and the helicoid $y(u, v) = (\sinh v \cos u, \sinh v \sin u, u)$ have the same Gauss curvature.
   (b) Show that $(\cos t)x + (\sin t)y$ is a one parameter family of minimal surfaces (i.e. $\mathrm{tr}(S) = 0$) and that they are all isometric to each other.