ASSIGNMENT # 6

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let $u, v, z \in T_pM$ and $f(t,s)$ a 2-parameter family with $f(0,0) = p,\ f_t(0,0) = u,\ f_s(0,0) = v$. Define $z(s_0) \in T_pM$ as the parallel translate of z around the closed curve which is the image under f of the boundary of the square $0 \leq t \leq s_0,\ 0 \leq s \leq s_e$. Show that

$$R(u,v)z = \lim_{s \to 0} \frac{z(s) - z}{s^2}$$

Hint: Define a vector field $Z(t,s)$ along f by parallel translating z first along the t parameter curve $s = 0$ and then along the s parameter curve to $f(t,s)$ and then consider $R(f_t, f_s)Z$.

(2) Let M^2 be a 2-dimensional Riemannian manifold.
   (a) Show that the conjugate points along a geodesic are isolated and have multiplicity one.
   (b) Let γ be a geodesic and assume that $\gamma(t_0)$ is conjugate to $\gamma(0)$ along γ. Show that nearby geodesics intersect γ near $\gamma(t_0)$ but after t_0.
   (c) Let $C_k = \{v \in T_pM \mid \exp(v)$ is the k-th conjugate point to p along the geodesic $t \to \exp(tv)\}$. Show that $C_k \subset T_pM$ is a smooth curve. Is $\exp(C_k)$ smooth?

(3) Let $\gamma(t),\ 0 \leq t \leq t_0$ be a geodesic without self intersections such that $\gamma(t)$ is not conjugate to $\gamma(0)$ for all $t \leq t_0$. Show that γ is locally minimizing, i.e. there exists a neighborhood U of $\mathrm{Im}(\gamma)$ such that any curve in U from $\gamma(0)$ to $\gamma(t_0)$ has length $\geq L(\gamma)$ and equal length iff it agrees with γ up to parametrization.

(4) If M has non-positive sectional curvature, show that $|d(\exp)_p)_v(w)| \geq |w|$ for all $v, w \in T_pM$. If we have equality, show that $\sec(\gamma'(t), E_w) = 0$ for all $t \leq 1$, where $\gamma(t) = \exp(tv)$ and E_w is the parallel vector field along γ with $E_w(0) = w$. Furthermore, E_w is also a Jacobi field.

(5) Use Jacobi fields to give a simple proof of the following facts:
   (a) All distance spheres $\partial B_r(p)$ are orthogonal to the geodesics starting at p.
   (b) The Gauss curvature of a metric in polar coordinates $ds^2 = dt^2 + f(t,\theta)^2 d\theta^2$ is given by $-f_{tt}/f$.