ASSIGNMENT # 7
MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let \( u, v, z \in T_pM \) and \( f(t, s) \) a 2-parameter family with \( f(0, 0) = p, \ f_t(0, 0) = u, \ f_s(0, 0) = v \). Define \( z(s_0) \in T_pM \) as the parallel translate of \( z \) around the closed curve which is the image under \( f \) of the boundary of the square \( 0 \leq t \leq s_0, \ 0 \leq s \leq s_c \). Show that
\[
R(u, v)z = \lim_{s \to 0} \frac{z(s) - z}{s^2}
\]
Hint: Define a vector field \( Z(t, s) \) along \( f \) by parallel translating \( z \) first along the \( t \) parameter curve \( s = 0 \) and then along the \( s \) parameter curve to \( f(t, s) \) and then consider \( R(f_t, f_s)Z \).

(2) Let \( M^2 \) be a 2-dimensional Riemannian manifold.
(a) Show that the conjugate points along a geodesic are isolated and have multiplicity one.
(b) Let \( \gamma \) be a geodesic and assume that \( \gamma(t_0) \) is conjugate to \( \gamma(0) \) along \( \gamma \). Show that nearby geodesics intersect \( \gamma \) near \( \gamma(t_0) \) but after \( t_0 \).
(c) Let \( C_k = \{ v \in T_pM \mid \exp(v) \) is the \( k \)-th conjugate point to \( p \) along the geodesic \( t \to \exp(tv) \} \). Show that \( C_k \subset T_pM \) is a smooth curve. Is \( \exp(C_k) \) smooth?

(3) Let \( \gamma(t), \ 0 \leq t \leq t_0 \) be a geodesic without self intersections such that \( \gamma(t) \) is not conjugate to \( \gamma(0) \) for all \( t \leq t_0 \). Show that \( \gamma \) is locally minimizing, i.e. there exists a neighborhood \( U \) of \( \text{Im}(\gamma) \) such that any curve in \( U \) from \( \gamma(0) \) to \( \gamma(t_0) \) has length \( \geq L(\gamma) \) and equal length iff it agrees with \( \gamma \) up to parametrization.

(4) Consider a product metric on \( M \times N \).
(a) Show that the sectional curvature of a 2-plane spanned by \( (X_1, X_2) \) and \( (Y_1, Y_2) \) is given by the mean value
\[
\frac{|X_1 \wedge Y_1|^2 \sec(X_1, Y_1) + |X_2 \wedge Y_2|^2 \sec(X_2, Y_2)}{|(X_1, Y_1) \wedge (X_2, Y_2)|^2}
\]
where \( \sec(X_1, Y_1) = 0 \) if \( X_1 \) and \( Y_1 \) are linearly dependent.
(b) Show that \( M \times N \) contains a flat totally geodesic surface.
(c) Show that the product metric on \( \mathbb{S}^n(r) \times \mathbb{S}^m(s) \) has \( \sec \geq 0 \) and \( \text{Ric} > 0 \) and becomes Einstein for appropriate \( r \) and \( s \).

(5) Describe the first conjugate locus to \( p \in M \), both as a subset of \( T_pM \) and of \( M \), for the product metric on \( \mathbb{S}^n(1) \times \mathbb{S}^m(1) \) for all \( n, m \geq 1 \).

(6) Use Jacobi fields to give a simple proof of the following facts:
(a) All distance spheres \( \partial B_r(p) \) are orthogonal to the geodesics starting at \( p \).
(b) The Gauss curvature of a metric in polar coordinates \( ds^2 = dt^2 + f(t, \theta)^2 d\theta^2 \) is given by \( -f_{tt}/f \).