ASSIGNMENT # 9

MATH 660 , DIFFERENTIAL GEOMETRY

(1) Recall that for a geodesic γ we have the index form

$$I(V,W) = \int_0^a \langle V', W' \rangle - \langle R(V,\gamma')\gamma', W \rangle dt$$

where V, W are proper vector fields along γ.

(a) Show that the kernel of I is given by the set of Jacobi fields vanishing at the endpoints.

(b) Show that if γ has no conjugate point, then I is positive definite.

(2) Let $N_1, N_2 \subset M^n$ be two compact totally geodesic submanifolds. Show that:

(a) If $\sec_M > 0$ and $\dim N_1 + \dim N_2 \geq \dim M$ show that N_1 and N_2 intersect.

(b) If N_1 and N_2 are minimal hypersurfaces and $\mathrm{Ric}_M > 0$, show that N_1 and N_2 intersect.

(3) Let $\gamma\colon [0,a] \to M$ be a geodesic and $L(c) = \int_0^a |c'(t)|dt$ be the length of a curve. Let c_s be a proper variation (i.e. end points fixed) with $c_0 = \gamma$, $V = \frac{d(c_s)}{ds}_{|s=0}$ its variational vector field, and $L(s) = L(c_s)$. Show that $L'(0) = 0$ and

$$L''(0) = \int_0^a \langle V_\perp', V_\perp' \rangle - \langle R(V_\perp, \gamma')\gamma', V_\perp \rangle dt$$

where $V_\perp = V - \langle V, \gamma' \rangle \gamma' / |\gamma'|^2$ is the component of V orthogonal to γ'.

(4) Fix $p \in M$, and for each unit vector $v \in \mathrm{T}_p M$ let $t(v)$ be the first conjugate value along the geodesic $t \to \exp(tv)$. Using the index form to show that t is continuous.

(5) Let M be a complete simply connected manifold with $\sec \leq 0$. In class we showed that if Δ is a triangle with angles α_i, then $\sum \alpha_i \leq 0$. In the case of equality, show that Δ is the boundary of a totally geodesic flat surface. You may want to use Problem 4 in Ass 6.