ASSIGNMENT # 10
MATH 660 , DIFFERENTIAL GEOMETRY

(1) Let \( N^k \subset M^n \) be a compact submanifold and
\[
\nu_\epsilon = \{ v \in T_pM \mid p \in N, \text{ and } v \perp T_pN \text{ with } |v| \leq \epsilon \}.
\]
Show that there exists an \( \epsilon > 0 \) such that \( \exp^\perp : \nu_\epsilon \to M \) (where \( \exp^\perp(v) = \exp_p(v) \) for \( v \in T_pM \)) is a diffeomorphism onto its image. In this case \( T_\epsilon(N) = \exp^\perp(\nu_\epsilon) \) is called an \( \epsilon \) tubular neighborhood of \( N \).

(2) \( N^k \subset M^n \) as above. Prove an analogue of the Gauss Lemma: The geodesic \( \exp^\perp(tv) \) meets \( \partial T_\delta(N) \) orthogonally for all \( \delta \leq \epsilon \).

(3) \( N^k \subset M^n \) as above. If \( \gamma(t) = \exp^\perp(tv), \ t \leq a \) is a geodesic in a tubular neighborhood, show that \( \gamma \) is the shortest connection from \( N \) to \( \gamma(a) \).

(4) Let \( N^k \subset R^N \) be a submanifold in Euclidean space. What are the focal points of \( N \) along a geodesic orthogonal to \( N \)?

(5) What is the analogue of the index form for curves starting at \( p \in M \) and ending at \( N \subset M \). Show that it is positive definite if there are no focal points.