ASSIGNMENT # 12
MATH 660, DIFFERENTIAL GEOMETRY

(1) Show that a Riemannian manifold on which the isometry group acts transitively is complete. Furthermore, show that every geodesic loop is a closed geodesic.

Hint for part 2: Prove and use the fact that the vector field associated to a one parameter group of isometries is a Jacobi field when restricted to any geodesic.

(2) Let $M$ be a compact even dimensional manifold with $0 < \sec \leq 1$. Show that the injectivity radius satisfies $i(M) \geq \pi/2$.

(3) Compute the cut locus of a lens space $S^3/Z_k$ where $Z_k \subset S^1 \subset \mathbb{C}$ acts via multiplication on each complex coordinate on $S^3 \subset \mathbb{C}^2$. Show in particular that it is independent of the point.

(4) Let $N \subset M^n$ be a hypersurface, $\gamma$ a normal geodesic starting orthogonal to $N$, and $t_0$ its first focal point. Furthermore, assume that $\text{Ric}_M \geq (n-1)\delta > 0$. If $N$ is minimal, show that $t_0 \leq \frac{\pi}{2\sqrt{\delta}}$. If it is not minimal, but has mean curvature $H$, show that $t_0 \leq \frac{1}{\sqrt{\delta}} \arctan(\sqrt{\frac{\delta}{H}})$.

(5) We showed in class that $\text{Ric} \geq (n-1)\delta$ implies that $\text{vol}(B_r(p)) \leq \text{vol}(B_r^\delta(p_o))$ where $B_r^\delta(p_o)$ is a ball in constant curvature $\delta$ space. Show that the analogous statement for $\text{Ric} \leq (n-1)\delta$ does not hold.

Hint: Fubini Study metric on $\mathbb{CP}^n$ and use the computation of the Jacobi fields from Problem (2) in Assignment #11.

(6) Let $\gamma$ be a geodesic in $M$. Show that along the geodesic $\gamma$, focal points come before conjugate points.

Left over from last 2 weeks

(7) Compute the cut locus of any(!) flat metric on the 2-torus and the Klein bottle.

(8) Show that the $n$-dimensional paraboloid $\{x_1, \ldots, x_n, \sum_i x_i^2\} \subset \mathbb{R}^{n+1}$ has positive sectional curvature.

(9) (generalized Morse Schoenberg) Let $M^n$ and $\tilde{M}^n$ be two manifolds with $\text{sec}_{\tilde{M}} \geq \text{sec}_M$ and $\gamma$ and $\tilde{\gamma}$ be two normal geodesics in $M$ resp. $\tilde{M}$ with $L(\gamma) = L(\tilde{\gamma})$. Show that $\text{ind}(\tilde{\gamma}) \geq \text{ind}(\gamma)$.

Use this result to prove both directions of Problem (6) in Assignment #11.

Left over from last week is also Problem (2) and (3) from Assignment #11.