6th Oct 2014 Riemannian submersions

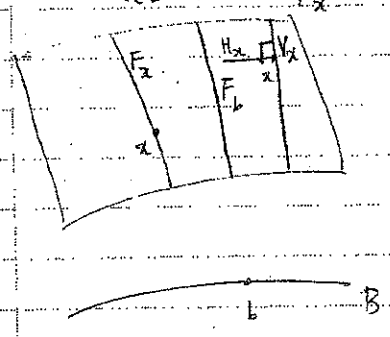
(Exercises in do Carmo, pg 185-190)

<u>submersion</u> $\equiv$ $\pi : M^{n+k} \longrightarrow B^n$ s.t. π_* is onto.

normal form: $\exists$ coordinates $(x_1, \ldots x_{n+k})$ for M and $(y_1, \ldots y_n)$ for B s.t. $\pi(x_1, \ldots x_{n+k}) = (y_1, \ldots y_n)$

$\Rightarrow$ $b \in B$ then $\pi^{-1}(b) = F_b$ is a submanifold (embedded)

$x \in M$ $\quad F_x = \pi^{-1}(\pi(x))$



vertical space $(x \in M)$: $V_x = T(F_x)$

if M and B has a Riemannian metric then define <u>horizontal space</u> : $H_x = V_x^{\perp}$

$d(\pi)_x : H_x \longrightarrow T_{\pi(x)} B$ is an <u>isomorphism</u>

<u>Defn:</u> π is a <u>Riemannian submersion</u> if this is an isometry.

$(M, \langle , \rangle)$ Riemannian manifold
define a distance function d:

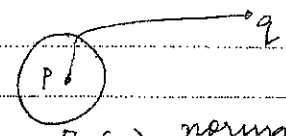
$$d(p,q) := \inf_{\substack{\gamma \text{ from} \\ p \text{ to } q}} L(\gamma)$$

$\therefore$ makes M into a metric space.

① $d(p,q) \geq 0$ $\qquad$ ② $d(p,q) = d(q,p$

3. $d(p,q)=0 \overset{?}{\Longrightarrow} p=q$

<u>proof</u>: (sketch)



$B_\varepsilon(p)$ normal ball s.t. $q \notin B_\varepsilon(p)$

then, any curve from p to q has length $\geq \varepsilon$

<u>Properties of Riemannian submersions</u>:

1) π decreases length. $d(x,y) \geq d(\pi(x), \pi(y))$

<u>proof</u>:

if γ is a curve in M then $L(\pi(\gamma)) \leq L(\gamma)$, since

$$L(\gamma) = \int |\gamma'|\, dt = \int |\gamma_V' + \gamma_H'|\, dt$$

$$\geq \int |\gamma_H'|\, dt$$

$$= \int |\pi(\gamma)'|\, dt$$

2) If X is a v.f. on B then there exists a unique v.f. $\overline{X}$ on M s.t. $\overline{X}(y) \in H_y$ all $y \in M$ and $d\pi(\overline{X}) = X$

($\overline{X}$ is a horizontal lift)

<u>define</u>: $\overline{X}(y) = (d\pi)|_{H_y}^{-1}\left(X(\pi(y))\right)$

$\overline{X}$ is smooth: locally, $\pi(x_1, \ldots x_{n+k}) = (y_1, \ldots y_n)$

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial y_i}, \quad \tilde{X} = \sum_{i=1}^{n} a_i \circ \pi \frac{\partial}{\partial x_i}$$

then $d\pi(\tilde{X}) = X$

(but $\tilde{X}$ need not be horizontal!)

so, use Gram-Schmidt to project $\tilde{X}(y)$ into H_y.

$V_y = \{ \frac{\partial}{\partial x_{n+1}}, \dots \frac{\partial}{\partial x_{n+k}} \}$

call this projection $\bar{X}$. Then $\bar{X}$ is smooth, & horizontal

3) If γ is a curve in B then $\exists$ a local lift $\bar{\gamma}$ in M s.t.
$\pi(\bar{\gamma}) = \gamma$, $\bar{\gamma}' \in H_{\gamma(t)}$

$\bar{\gamma}$ is called the horizontal lift of γ.

assume $\gamma'(t) \neq 0 \ \forall t$
$\mathrm{Im}(\gamma) \subset B$ is a submanifold.

then $L := \pi^{-1}(\mathrm{Im}(\gamma))$ is a submanifold of M
and on L $\exists$ a v.f. Z which is a horizontal lift
of v.f. γ' on L

$\pi : L \longrightarrow \mathrm{Im}(\gamma)$ is a Riemannian submersion.

integral curves of Z are the lifts $\bar{\gamma}$ that we
want to construct.
$Z \longleftarrow \gamma'$ integral curves exist locally.

and $\bar{\gamma}$ exists globally if M is metrically complete
(M, d).
[$\Longleftrightarrow$ M is geodesically complete, i.e. all geodesics
are defined for all t]

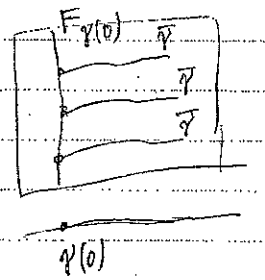
Hopf-Rinow.

If (M,d) metrically complete. lift $\bar{\gamma}$ is defined on $[0,t_0)$

$\gamma(t_i)$ is a Cauchy sequence. $d(\gamma(t_i), \gamma(t_j)) \le L(\gamma|_{[t_i,t_j]})$

$\Rightarrow$ subseq $\gamma(t_i) \longrightarrow p \in M$, v.f. Z is defined at p has a local flow.

the lift $\bar{\gamma}$ is unique once I specify beginning point $\bar{\gamma}(0) \in \pi^{-1}(\gamma(0))$



4) If γ is a geodesic in $B \Rightarrow \bar{\gamma}$ is a geodesic in M.

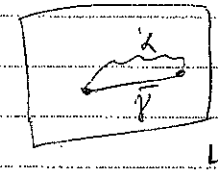
$L(\gamma) = L(\bar{\gamma})$

$\bar{\gamma}'(t) \in H_{\bar{\gamma}(t)}$ and that γ is locally minimizing.

$|\gamma'(t)| = |\bar{\gamma}'(t)|$

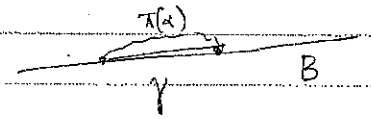
$\Rightarrow \bar{\gamma}$ is also locally minimizing since π decreases distance.

$\Rightarrow \bar{\gamma}$ is a geodesic.



$|\bar{\gamma}'(t)| = |\gamma'(t)|$ is constant.

$L(\pi(\alpha)) \le L(\alpha)$.



5) If γ is a geodesic in M s.t. $\gamma'(0) \in H_{\gamma(0)}$ then $\gamma'(t) \in H_{\gamma(t)}$ $\forall t$ and $\pi \circ \gamma$ is a geodesic in B.

Let δ be the geodesic in B with $\delta(0) = \pi(\gamma(0))$
$\delta'(0) = d\pi(\gamma'(0))$

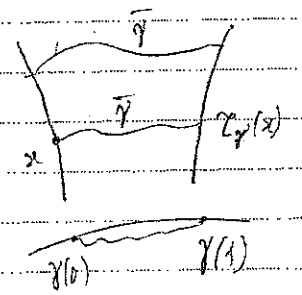
Let $\bar{\delta}$ be the horizontal lift of δ, is a geodesic in M (by part (4))
s.t. $\bar{\delta}(0) = \gamma(0)$, $\bar{\delta}'(0) = \gamma'(0)$.
$\Rightarrow \gamma = \delta$ by uniqueness of geodesics.

hence, if M is geodesically complete $\Rightarrow$ B is geod. complete.

6) holonomy: let $\gamma \subset B$ ($\gamma : [0,1] \longrightarrow B$) be a path from p to q.

$$\tau_\gamma : F_{\gamma(0)} \longrightarrow F_{\gamma(1)}$$

$$x \longmapsto \bar{\gamma}(1) \text{ where } \gamma(0) = x$$



($\bar{\gamma}$ is a horizontal lift.)

$$(\tau_\gamma)^{-1} = \tau_{\gamma^{-1}}$$

(this is a diffeomorphism b/w the two fibers)

7) If M is complete then π is a locally trivial fiber bundle with fiber F
$(\pi^{-1}(U) \simeq U \times F_p)$

choose U to be a normal nbhd of p.

let $q \in U$. $\exists$ unique minimizing geod. γ_{pq} which defines a diffeo b/w F_q and F_p.

8) Fibers are constant distance apart.

fibers F_p, F_q $\quad x \in F_p$. $\quad d(x, F_q) := \inf_{y \in F_q} d(x,y)$

$$\boxed{d(y, F_q) = d(y', F_q) \quad \forall\ y, y' \in F_p} \qquad (*)$$

<u>Exercise</u>: Prove the converse. i.e. if condition $(*)$ holds, show that it's a Riemannian submersion.

<u>Proof of $(*)$</u>:

Choose γ a geod. in B from p to q s.t. $L(\gamma) = d(p,q)$

<u>show</u>: $d(y', F_q) \le d(y, F_q)$ (by symmetry they must be same)

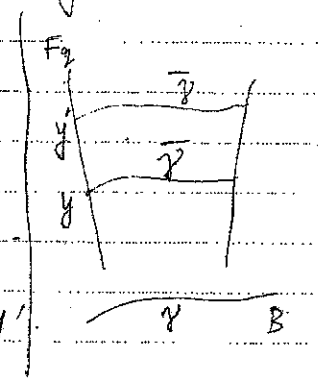
Let γ_i be curves in M from y to F_q

$L(\gamma_i) \longrightarrow d(y, F_q)$.

then $L(\pi \circ \gamma_i) \le L(\gamma_i)$ $\qquad \gamma_i(0) = y$.

lift $\pi \circ \gamma_i$ horizontal.

$\overline{\pi \circ \gamma_i}$ $\subset M$. $\qquad \overline{\pi \circ \gamma_i}(0) = y'$.



$L(\overline{\pi \circ \gamma_i}) \le L(\gamma_i)$

$\longrightarrow$ $\square$

<u>Examples</u>:

1) $B \times F$ product metric $\qquad B \times F \xrightarrow{\pi_1} B$

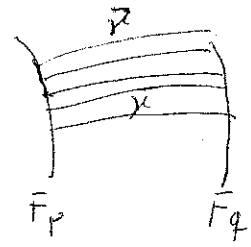
2) warped products.

$f \in C^\infty(B)$, $\langle , \rangle_B$, $\langle , \rangle_F$ define metric on $B \times F$ by:

9) The fibers of π are totally geodesic iff the holonomy diffeomorphisms $\tau_\gamma : F_p \to F_q$ $\gamma(0)=p$ $\gamma(1)=q$ are isometries

<u>Proof</u> Let $c : (-\varepsilon,\varepsilon) \to F_p$ be a curve with $v = c'(0) \in TF_p$

define variation $c(s,t) = \tau_{\gamma(t)}(c(s))$ where $\gamma(t) = \gamma_{|[0,t]}$



then $\frac{\partial}{\partial t}\langle \frac{\partial c}{\partial s}, \frac{\partial c}{\partial s}\rangle = 2\langle \frac{D}{dt}\frac{\partial c}{\partial s}, \frac{\partial c}{\partial s}\rangle$

$= 2\langle \frac{D}{ds}\frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\rangle - 2\langle \frac{\partial c}{\partial t}, \frac{D}{ds}\frac{\partial c}{\partial s}\rangle$

since $\frac{\partial c}{\partial t}$ is normal to all fibers and $\frac{\partial c}{\partial s}$ tangent

$= -2\langle \frac{\partial c}{\partial t}, \nabla_{\frac{\partial c}{\partial s}}\frac{\partial c}{\partial s}\rangle = -2\langle \frac{\partial c}{\partial t}, B(\frac{\partial c}{\partial s}, \frac{\partial c}{\partial s})\rangle$

But $\frac{\partial c}{\partial s} = d(\tau_{\gamma(t)})(\frac{\partial c}{\partial s}) = d(\tau_{\gamma(t)})(v)$

Thus $\frac{\partial c}{\partial s}$ has contant length iff $\langle \frac{\partial c}{\partial t}, B(\frac{\partial c}{\partial s}, \frac{\partial c}{\partial s})\rangle = 0$

If this is true for all γ (hence any $\frac{\partial c}{\partial t}$) and for all $c(s)$ (hence any $\frac{\partial c}{\partial s}$) then $B \equiv 0$

i.e. fibers are tot geodesic

<u>Cor</u> If all fibers are totally geodesic, then they are all isometric to each other

$$\langle\, , \rangle = \langle\, , \rangle_B + f(p) \langle\, , \rangle_F$$

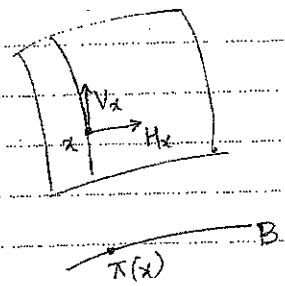
for $(p, f) \in B \times F$ | $B \times F \xrightarrow{\pi} B$, $TB \perp TF$

or e.g. $dt^2 + f^2(t)\, d\theta^2$ polar coordinates.
surface of revolution.

$$\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$dt \quad d\theta$$

3) Let G be a Lie group acting on $(M, \langle\, , \rangle)$ by isometries and acting freely.
M/G is a manifold. $M \xrightarrow{\pi} M/G = B$

$\exists$ a metric on B s.t. π is a Riemannian submersion.



$d\pi|_{H_x} : H_x \longrightarrow T_{\pi(x)} B$ is an isom.

choose metric on $T_{\pi(x)} B$ s.t. $d\pi|_{H}$ is an isometry.

This metric on B is well-defined (independent of choice of ~~x~~ point in $\pi^{-1}(\pi(x))$:
say $x, y \in \pi^{-1}(\pi(x))$. $\Rightarrow \exists\, g \in G$ s.t. $y = gx$. and
dL_g takes vertical space to vertical space.
L_g preserves an orbit.

dL_g takes H_x isom into H_y.
$dL_g \cdot d\pi(v) = d\pi(v)$ $\quad L_g \circ \pi = \pi$.
$v \in H_x$

e.g. $\mathbb{C}P^n$ = complex lines in $\mathbb{C}^{n+1}$
$\cup$
S^{2n+1}

$S^1 \circlearrowright S^{2n+1}$: $e^{i\theta} \cdot (z_1, \ldots z_{n+1}) = (e^{i\theta} z_1, \ldots e^{i\theta} z_{n+1})$

this is an isometric action.

$$S^{2n+1}/S^1 = \mathbb{C}P^n.$$

induces metric on $\mathbb{C}P^n$, called Fubini-Study metric.

7th Oct 2014

Riemannian submersions

recall,

$$\pi : (M, \langle , \rangle) \longrightarrow (B, \langle , \rangle)$$

submersion ($d\pi$ onto)

$x \in M$ $\quad V_x = T(\underbrace{\pi^{-1}(\pi(x))}_{F_x})$ vertical space

$$H_x = V_x^{\perp} \quad \text{horizontal space}$$

π is called a Riemannian submersion if $d\pi : H_x \longrightarrow T_{\pi(x)}B$ is an isometry.

Properties.

1) $d(\pi(x), \pi(y)) \leq d(x,y)$

2) X v.f. on B, there exists horizontal lift $\overline{X}$ v.f. on M s.t. $\overline{X}(y) = H_y$ and $d\pi(\overline{X}) = X$.

3) If γ is a curve in B (piecewise regular) then $\exists$ horizontal lift $\overline{\gamma}$.

$\pi(\bar\gamma) = \gamma$ and $\bar\gamma'(t) \in H_{\bar\gamma(t)}$

$\bar\gamma$ is unique if we choose $\bar\gamma(0) \in \pi^{-1}(\gamma(0))$

(exists locally and $\forall t$ if M is complete)

4) If γ is a geodesic in B then $\bar\gamma$ is a geod. in M.

5) If c is a geod. in M s.t. $c'(0) \in H_{c(0)} \Rightarrow c'(t) \in H_{c(t)}$
$\Rightarrow \pi(c)$ is a geodesic.

6) hol. diffeo. If γ curve in B then

$$\tau_\gamma : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(1)) \text{ diffeo.}$$
$$x \longmapsto \bar\gamma(1) \text{ where } \bar\gamma(0) = x.$$

7) π is locally trivial.

<u>O'Neill formula</u>

$$\pi : M \longrightarrow B$$
$$\nabla^M \qquad \nabla^B$$

<u>claim</u>: $\langle \nabla^B_X Y, Z \rangle = \langle \nabla^M_{\bar X} \bar Y, \bar Z \rangle$

X, Y, Z v.f. on B and $\bar X, \bar Y, \bar Z$ their horizontal lifts.

<u>proof</u>: $\langle\ ,\ \rangle := \langle\ ,\ \rangle$
Check it satisfies Levi-Civita properties

a) product rule
$$X\langle Y, Z\rangle = \bar X \langle \bar Y, \bar Z\rangle$$

$$\langle Y, Z\rangle = \langle \bar Y, \bar Z\rangle$$

let $\gamma(t)$ be ~~the~~ integral curve of X

$\langle , \rangle$ $\qquad$ $\bar{\gamma}$ integral curve of $\bar{X}$

$$\langle Y, Z\rangle_{\gamma(t)} = \langle \bar{Y}, \bar{Z}\rangle_{\bar{\gamma}(t)}$$

b) torsion free

$$\langle [X,Y], Z\rangle = \langle \nabla^B_X Y, Z\rangle - \langle \nabla^B_Y X, Z\rangle$$

$$= \langle \nabla^M_{\bar{X}} \bar{Y}, \bar{Z}\rangle - \langle \nabla^M_{\bar{Y}} \bar{X}, \bar{Z}\rangle$$

$$= \langle [\bar{X}, \bar{Y}], \bar{Z}\rangle \underset{?}{=} \langle [X,Y], Z\rangle$$

$$\pi_*([\bar{X}, \bar{Y}]) = [\pi_* \bar{X}, \pi_* \bar{Y}]$$

$$= [X, Y]$$

$$\pi_*(\bar{Z}) = Z$$

$$\overline{[X,Y]} = \mathcal{H}[\bar{X}, \bar{Y}]$$ $\mathcal{H}$ ← means take horizontal component.

hor + vert.

$$\nabla^M_{\bar{X}} \bar{Y} = \underbrace{\nabla^B_X Y}_{\text{hor. comp.}} + \underbrace{A_X Y}_{\text{vertical component.}}$$

A O'Neil tensor

$$A_X Y = (\nabla^M_{\bar{X}} \bar{Y})^v$$ ← means project onto vertical space.

$$\| $$

$$A_{\bar{X}} \bar{Y}$$

<u>Lemma</u>: (a) A is a tensor

(b) $A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ bilinear, skew $(A_X Y = -A_Y X)$

(c) $A_X Y = \frac{1}{2}[\overline{X}, \overline{Y}]^{\mathcal{V}}$

<u>Cor</u>: $A = 0$ <u>iff</u> horizontal distribution is integrable

<u>Proof of lemma</u>:

(a) linear over C^∞.

$$\overline{fX} = f \circ \pi \, \overline{X}$$

linearity in X is clear.

for linearity in Y:

$$A_X(fY) = \left(\nabla^M_{\overline{X}} \overline{(fY)}\right)^{\mathcal{V}} = \left(\nabla^M_{\overline{X}} (f \circ \pi)\overline{Y}\right)^{\mathcal{V}}$$

$$= \Big(\underbrace{\overline{X}(f \circ \pi) \cdot \overline{Y}}_{\text{this term goes away because we're taking vertical component}} + f \circ \pi \, \nabla^M_{\overline{X}} \overline{Y}\Big)^{\mathcal{V}}$$

$$= f \circ \pi \cdot \left(\nabla^M_{\overline{X}} \overline{Y}\right)^{\mathcal{V}} = f \cdot A_X Y$$

(b) $A_X Y = -A_Y X$ <u>iff</u> $A_X X = 0$.

$$\langle A_X X, U \rangle = \langle \nabla^M_{\overline{X}} \overline{X}, U \rangle$$

(U is a v.f on M that is vertical : $U(x) \in V_x$)

$$= \underbrace{\overline{X}\langle \overline{X}, U \rangle}_{=0} - \langle \overline{X}, \nabla^M_{\overline{X}} U \rangle$$

$$\overset{!}{=} -\langle \overline{X}, \nabla^M_U \overline{X} \rangle = \frac{1}{2} U\left(\langle \overline{X}, \overline{X} \rangle\right) = 0$$

$\langle \bar{X}, \bar{X} \rangle = \langle X, X \rangle$ is constant along a fiber.

<u>Proof of !</u> it's equivalent to:

$\langle \bar{X}, [\bar{X}, u] \rangle = 0$

$[\bar{X}, u] \in \mathcal{V}$?

$$\pi_* [\bar{X}, u] = [\pi_* \bar{X}, \underset{0}{\underset{\shortparallel}{\pi_* u}}] = 0$$

$$\Rightarrow [\bar{X}, u] \in \mathcal{V}.$$

(e) $[\bar{X}, \bar{Y}] = \nabla^M_{\bar{X}} \bar{Y} - \nabla^M_{\bar{Y}} \bar{X}$

$$[\bar{X}, \bar{Y}]^{\mathcal{V}} = A_X Y - A_Y X = 2 A_X Y$$

Riemannian submersions
(formula for curvature using O'Neill tensor)

Notation: $\pi : M \longrightarrow B$ is a Riemannian submersion.
∇^M = connection of M
∇^B = connection of B
R^M = curvature tensor of M
R^B = curvature tensor of B

for a vector field X on M, $X^{\mathcal{H}}$ refers to its horizontal component and $X^{\mathcal{V}}$ its vertical component.

O'Neill tensor: let X, Y be vector fields on B, $\overline{X}, \overline{Y}$ their horizontal lifts (vector fields on M). Then we define a tensor A by:

$$A_X Y = \left(\nabla^M_{\overline{X}} \overline{Y}\right)^{\mathcal{V}}$$

Then the ~~vector fi~~ curvature tensors of M and B are related by the following proposition:

Propn. $\langle R^M(\overline{X}, \overline{Y})\overline{Y}, \overline{X}\rangle = \langle R^B(X,Y)Y, X\rangle - 3\,|A_X Y|^2$

Proof. Step 1:
let X, Y, Z, W be vector fields on B and $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ their horizontal lifts. Then:

$$\langle \nabla^M_{\overline{X}} \nabla^M_{\overline{Y}} \overline{Z}, \overline{W}\rangle = \langle \nabla^M_{\overline{X}} (\overline{\nabla^B_Y Z}), \overline{W}\rangle + \langle \nabla^M_{\overline{X}} (A_Y Z), \overline{W}\rangle$$

$$= \langle \overline{\nabla^B_X \nabla^B_Y Z} + A_X \nabla^B_Y Z, \overline{W}\rangle - \langle A_Y Z, \nabla^M_{\overline{X}} \overline{W}\rangle$$

since $\langle A_Y Z, \overline{W}\rangle = 0$

$$= \langle \overline{\nabla^B_X \nabla^B_Y Z}, \overline{W}\rangle + \underbrace{\langle A_X \nabla^B_Y Z, \overline{W}\rangle}_{=0} - \underbrace{\langle A_Y Z, \overline{\nabla^B_X W}\rangle}_{=0} - \langle A_Y Z, A_X W\rangle$$

(inner product of vertical vector with horizontal

So,

$$\boxed{\langle \nabla^M_{\bar{X}} \nabla^M_{\bar{Y}} \bar{Z}, \bar{W} \rangle = \langle \nabla^B_X \nabla^B_Y Z, W \rangle - \langle A_Y Z, A_X W \rangle} \quad (1)$$

Step 2:

Now, curvature in M:

$$\langle R^M(\bar{X}, \bar{Y})\bar{Y}, \bar{X} \rangle := \langle \nabla^M_{\bar{X}} \nabla^M_{\bar{Y}} \bar{Y}, \bar{X} \rangle - \langle \nabla^M_{\bar{Y}} \nabla^M_{\bar{X}} \bar{Y}, \bar{X} \rangle - \langle \nabla^M_{[\bar{X},\bar{Y}]} \bar{Y}, \bar{X} \rangle$$

By (1), we can write this as:

$$= \langle \nabla^B_X \nabla^B_Y Y, X \rangle - \langle \underbrace{A_Y Y}_{=0}, \underbrace{A_X X}_{=0} \rangle - \langle \nabla^B_Y \nabla^B_X Y, X \rangle + \langle A_X Y, A_Y X \rangle$$
$$- \langle \nabla^M_{[\bar{X},\bar{Y}]} \bar{Y}, \bar{X} \rangle$$

$$\boxed{= \langle \nabla^B_X \nabla^B_Y Y, X \rangle - \langle \nabla^B_Y \nabla^B_X Y, X \rangle - |A_X Y|^2 - \langle \nabla^M_{[\bar{X},\bar{Y}]} \bar{Y}, \bar{X} \rangle} \quad (2)$$

Step 3:

$$\langle \nabla^M_{[\bar{X},\bar{Y}]} \bar{Y}, \bar{X} \rangle = \langle \nabla^M_{[\bar{X},\bar{Y}]^{\mathcal{H}}} \bar{Y}, \bar{X} \rangle + \langle \nabla^M_{[\bar{X},\bar{Y}]^{\mathcal{V}}} \bar{Y}, \bar{X} \rangle$$

$$= \langle \nabla^B_{[X,Y]} Y, X \rangle + \langle \nabla^M_{2A_X Y} \bar{Y}, \bar{X} \rangle$$

$$\boxed{= \langle \nabla^B_{[X,Y]} Y, X \rangle + 2 \cdot \langle \nabla^M_{A_X Y} \bar{Y}, \bar{X} \rangle} \quad (3)$$

Now, $\nabla^M_{A_X Y} \bar{Y} - \nabla^M_{\bar{Y}} A_X Y = [A_X Y, \bar{Y}]$ = vertical vector field

(bracket of vertical & horizontal is vertical)

$$\Rightarrow \langle \nabla^M_{A_X Y} \bar{Y}, \bar{X} \rangle = \langle \nabla^M_{\bar{Y}} A_X Y, X \rangle$$

$$= -\langle A_X Y, \nabla^M_{\bar{Y}} \bar{X} \rangle$$

$$= -\langle A_X Y, A_Y X \rangle = |A_X Y|^2$$

substituting this in ③, we get:

$$\boxed{\langle \nabla^M_{[\bar{X},\bar{Y}]} \bar{Y}, \bar{X} \rangle = \langle \nabla^B_{[X,Y]} Y, X \rangle + 2|A_X Y|^2} \quad (4)$$

substituting ④ in ~~③~~ ② we get:

$$\langle R^M(\bar{X},\bar{Y})\bar{Y}, \bar{X} \rangle$$
$$= \langle \nabla^B_X \nabla^B_Y Y, X \rangle - \langle \nabla^B_Y \nabla^B_X Y, X \rangle - |A_X Y|^2 - \langle \nabla^B_{[X,Y]} Y, X \rangle - 2|A_X Y|^2$$

$$= \langle R^B(X,Y)Y, X \rangle - 3|A_X Y|^2$$

so

$$\boxed{\langle R^M(\bar{X},\bar{Y})\bar{Y}, \bar{X} \rangle = \langle R^B(X,Y)Y, X \rangle - 3|A_X Y|^2}$$

$\square$

<u>Cor.</u> $\sec_B(X,Y) \geq \sec_M(\bar{X},\bar{Y})$

(Riemannian submersions increase curvature.)

See notes on homepage. : Fatness revisited.

$\pi : M \longrightarrow B$ is fat if $\pi^{-1}(b)$ are totally geodesic and $\sec(X, U) = |(\nabla_X U)^{\mathcal{H}}|^2 > 0$

$$\therefore \cancel{\langle R^M(\bar{X}, \bar{Y})\bar{Y}, X\rangle = \langle \nabla_X \nabla_Y Y, X\rangle} \quad \langle \nabla_Y \nabla_X Y, X\rangle - \langle \nabla_{[X,Y]} Y, X\rangle$$

$$+ 2\langle A_X Y, A_Y X\rangle + \langle A_X Y, A_Y X\rangle$$

$$= \langle R^B(X,Y)Y, X\rangle + 3 \cdot \langle A_X Y, A_Y X\rangle$$

$$= \langle R^B(X,Y)Y, X\rangle - 3 \cdot |A_X Y|^2$$

$\square$

<u>Example</u>: $\mathbb{C}P^n$ Fubini – Study metric.

$S^{2n+1}(1) \subset \mathbb{C}^{n+1}$

$$\pi : S^{2n+1}(1) \longrightarrow \mathbb{C}P^n$$
$$v \longmapsto [v] = \{\lambda v \mid \lambda \in \mathbb{C}\}$$

action of $S^1 = \{e^{i\theta}\} \subset \mathbb{C}$ on S^{2n+1}.

$$S^1 \times S^{2n+1} \longrightarrow S^{2n+1}$$
$$(e^{i\theta}, v) \longmapsto e^{i\theta} \cdot v$$

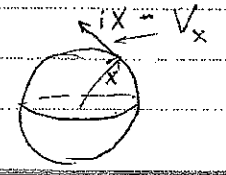
$v, w \in S^{2n+1}(1)$ $\quad \pi(v) = \pi(w) \Rightarrow v = e^{i\theta} w$ for some $e^{i\theta} \in S^1$

$S^{2n+1}(1)/S^1 = \mathbb{C}P^n$

action of S^1 is isometric $\Rightarrow$ π is a Riemannian submersion.

$x \in S^{2n+1}(1)$. $\quad V_x$ spanned by $\left.\frac{d}{d\theta}\right|_{\theta=0} (e^{i\theta} X) = i \cdot X$

$H_x = \{x, ix\}^{\perp}$ inside $\mathbb{R}^{2n+2}$



$$T_{[\bar{x}]}\mathbb{C}P^n = [(x,v)] \qquad (x,v)\sim(e^{i\theta}x, e^{i\theta}v)$$

$v \in H_x$; $v \perp x$, $v \perp ix$.

<u>geodesics in $\mathbb{C}P^n$</u>

$$\gamma_v(t) \qquad |v| = 1.$$

$$\gamma_v(0) = x \qquad , v \perp x, ix$$

$$\gamma_v(t) = [(\cos t)x + (\sin t)v]$$ equiv. class of great circle

all geodesics are closed of length π.

$$\gamma_v(\pi) = [-x] = [x]$$

and they don't close earlier.

<u>proof</u>:

suppose : $\gamma_v(t) = \gamma_v(0) = [x] = [\cos t x + \sin t\, v] \quad t < \pi$

$$\Rightarrow e^{i\theta}x = (\cos t)x + (\sin t)v \text{ for some } \theta.$$

$$0 = \langle e^{i\theta}x, v\rangle = \sin t \langle v, v\rangle \neq 0.$$

$$\boxed{\text{Exercise: } \mathbb{C}P^1 = S^2\left(\tfrac{1}{2}\right)}$$

What is $A_X Y$?

<u>Lemma</u>: $[x] \in \mathbb{C}P^n$, $v, w \in H_x$ $\quad v, w \perp x, ix$

$$A_v w = \langle iv, w\rangle iX$$

Cor: Assume v, w o.n.

if $w \perp iv \Rightarrow \sec(v,w) = 1$

if $w = \pm iv \Rightarrow \sec(v,w) = 4$

14th Oct 2014

recall. $\pi : M^{n+k} \longrightarrow B^n$ Riemannian submersion.

$\mathcal{V}_x = T_x(\pi^{-1}(b))$

$H_x = \mathcal{V}_x^{\perp}$ $\pi_* : H_x \longrightarrow T_{\pi(x)}B$ is a geodesic on B.

γ a geod. on M s.t. $\gamma'(0) \in H_{\gamma(0)} \Rightarrow \gamma'(t) \in H_{\gamma(x)}$

and $\pi \circ \gamma$ is a geod. in B.

O'Neil tensor: $A : H \times H \longrightarrow V$

$$A_X Y = \left(\nabla^M_{\bar{X}} \bar{Y}\right)^{\mathcal{V}} = \frac{1}{2}\left[\bar{X}, \bar{Y}\right]^{\mathcal{V}}$$

O'Neil formula:

$$\sec_B(X,Y) = \sec_M(\bar{X}, \bar{Y}) + 3\|A_X Y\|^2$$

$$\geq \sec_M(\bar{X}, \bar{Y})$$

$\mathbb{C}P^n$ Fubini-Study metric.

general fact: If G acts isometrically and freely on M, then $\exists$ metric on $B = M/G$ s.t. $\pi : M \longrightarrow M/G$ is a R.S.

$\mathbb{C}P^n \quad S^{2n+1}(1) \subset \mathbb{C}^{n+1} \quad ; \quad S^{2n+1}/S^1 = \mathbb{C}P^n$

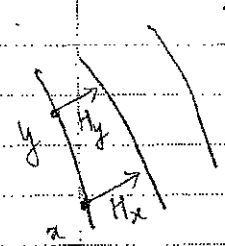
$$S^1 \times S^{2n+1} \longrightarrow S^{2n+1}$$
$$(e^{i\theta}, v) \longmapsto e^{i\theta} v \quad \| \text{ isometric action.}$$

$\pi : S^{2n+1}(1) \longrightarrow \mathbb{C}P^n$ gives a metric on $\mathbb{C}P^n$

<u>geodesics</u>: $x \in S^{2n-1} \quad [x] \in \mathbb{C}P^n$

$$T_{[x]}\mathbb{C}P^n = [(x, v)] \qquad v \perp x, ix$$

$$H_x = \{v \mid v \perp ix, v \perp ix\}$$



$$(x, v) \sim (e^{i\theta}x, e^{i\theta}v)$$

$y = e^{i\theta}x$

$H_y = e^{i\theta}H_x$

$$v \in T_{[x]}\mathbb{C}P^n$$

great circle in S^{2n+1}

$$\gamma(t) = x\cos t + v \sin t \in S^{2n+1} \quad \text{(geodesic)}$$

$$\gamma'(t) \in H_{\gamma(t)}$$

$$\gamma'(t) \perp \gamma(t) \text{ and } i.\gamma(t)$$

$$\langle -x\sin t + v\cos t, x\cos t + v\sin t\rangle = 0$$

$$\langle -x\sin t + v\cos t, ix\cos t + iv\sin t\rangle = -\overbrace{\langle x, ix\rangle}^{=0}\sin t\cos t + \underbrace{\langle v, iv\rangle}_{=0}\sin t\cos t$$

$$= 0$$

<u>Lemma</u>: $v, w \in H_x$. $A_v w = \langle iv, iw \rangle i x$ (O'Neil computations for $\mathbb{C}P^n$)

<u>Proof</u>: $A_v w = \left(\nabla^{S^{2n+1}}_{\bar{v}} \bar{w}\right)^{v}$

$$\gamma(t) = x\cos t + v \sin t \quad ; \quad [\gamma]'(0) = v.$$

$$\mathbb{R}^{2n+2} \ni w(t) = w\cos t + z \sin t \quad , \quad \text{some } z.$$

$$w(t) \in H_{\gamma(t)}$$

$[w]$ is a vector field along $[\gamma(t)]$

<u>one condition</u>:

$$0 = \langle w(t), \gamma(t) \rangle = \langle w\cos t + z\sin t, x\cos t + v \sin t \rangle$$

$$= \langle w, v \rangle \cos t \sin t + \langle z, x \rangle \sin t \cos t$$

$$+ \langle z, v \rangle \sin^2 t$$

need $\boxed{\begin{array}{l} \langle z, v \rangle = 0 \\ \langle z, x \rangle = -\langle w, v \rangle \end{array}}$

<u>another condition</u>:

$$0 = \langle w(t), i\gamma(t) \rangle = \langle w\cos t + z \sin t, ix\cos t + iv \sin t \rangle$$

$$= \langle w, iv \rangle \cos t \sin t + \langle z, ix \rangle \sin t \cos t$$

$$+ \langle z, iv \rangle \sin^2 t$$

$\therefore$ get additional conditions: $\boxed{\begin{array}{l} \langle z, iv \rangle = 0 \\ \langle z, ix \rangle = -\langle w, iv \rangle \end{array}}$

So, $\nabla_{\bar{v}} \bar{w} = \left(\nabla_v w\right)^{v} = v\left(w'(0)\right)$

$= v(z) = \langle z, ix\rangle ix = -\langle w, iv\rangle ix$

$= -\langle iw, i^2 v\rangle ix = \langle v, iw\rangle ix$

$\square$

<u>recall</u>: $\sec_B(v,w) = \sec_M(\bar{v},\bar{w}) + 3\|A_v w\|^2$ if v,w o.n.

So we get a corollary of the above lemma:

<u>Cor.</u> (let $v,w \in H_x$) $\sec_{\mathbb{C}P^n}(v,w) = \sec_{S^{2n+1}(1)}(v,w) + 3\|\langle v, iw\rangle ix\|^2$

$= 1 + 3\langle v, iw\rangle^2$

So $1 \leq \sec_{\mathbb{C}P^n} \leq 4$, and:

$\sec(v,w) = 1$ <u>iff</u> $v \perp iw$ "real plane"

$\sec(v,w) = 4$ <u>iff</u> $v = \pm iw$ "complex plane"

<u>Exercise</u>. totally geodesic submanifolds of $\mathbb{C}P^n$

$V \subset T_{[x]}\mathbb{C}P^n$. V is called complex if $iV = V$

V is called totally real if $V \perp iV$

- V is complex $\Rightarrow$ $\exp_{[x]}(V) = \mathbb{C}P^k \subset \mathbb{C}P^n$ tot. geod.

metric on $\mathbb{C}P^k$ Fubini-Study

- V is totally real $\Rightarrow \exp_{[x]} V = \mathbb{R}P^k(1) \subset \mathbb{R}P^n \subset \mathbb{C}P^n$ real pt.s.

- complex structure on M^{2n}

$$J : T_pM \longrightarrow T_pM \text{ linear and } J^2 = -Id$$

J is integrable if J is multiplication by i

complex mfd M, locally $\mathbb{C}^n$, interchanges hol.

<u>Theorem</u>. J is "integrable" iff ~~$N_J(X,Y)=0$~~

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] = 0$$

<u>Defn.</u> $(M, J, \langle,\rangle)$. $\langle,\rangle$ is <u>Hermitian</u> if $\langle Jv, Jw\rangle = \langle v,w\rangle$
($\Rightarrow$ J is skew since $J^2 = -Id$)

$$\langle Jv, w\rangle = \langle -v, Jw\rangle \qquad Jv \perp v.$$

<u>Defn.</u> $(M, J, \langle,\rangle)$ is called <u>Kähler</u> if Hermitian and $\nabla J = 0$
($\underset{\text{(Exercise)}}{\Longleftrightarrow}$ if V is parallel along γ then JV is also parallel)

$$(\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y)$$

- J is almost complex & $\nabla J = 0 \Rightarrow J$ is complex

<u>Defn.</u> Kähler form: $\omega\langle v, w\rangle = \langle Jv, w\rangle = -\langle v, Jw\rangle$

(ω is a 2-form)

$$\nabla J = 0 \iff d\omega = 0$$

Exercise. $\mathbb{C}P^n$ with Fubini-Study metric is Kähler. ($\nabla J = 0$)

Defn. holomorphic curvature: $\sec(v, iv) = \sec_{\mathbb{C}}$

e.g. $\mathbb{C}P^n$, $\sec_{\mathbb{C}} \equiv 1$ (and conversely)

($\mathbb{C}P^n$ is characterized by this property)

Defn. bisectional curvature: v, w.

$$\sec(v,w) + \sec(v, Jw) = \langle R(X, JX)Y, JY \rangle$$

Theorem. Frankel conjecture (Siu-Yau)
If M is compact $\pi_1 = 0$ Kähler + positive holomorphic bisectional curvature
$\Rightarrow$ biholomorphic to $\mathbb{C}P^n$.

Hopf-Rinow

Theorem: (a) M is metrically complete (for distance d) iff it is geodesically complete.

(b) if complete then any 2 points can be connected by a minimal geod. $L(\gamma) = d(\gamma(0), \gamma(t))$

(c) $\exp_p(\overline{B_r(0)}) = \overline{B_r(p)}$

(d) if $\exp_p$ is defined on T_pM for some p then also for all p.