**Riemannian submersions**

(Exercise in de Carmo, pg. 185-190)

**submersion**: \( \pi : M^{n+k} \rightarrow B^n \) s.t. \( \pi \) is onto.

normal form: \( \exists \) coordinates \((y_1, \ldots, y_n)\) for \(B\) s.t. \( \pi(y_1, \ldots, y_n) = (y_1, \ldots, y_n) \)

- \( \pi^{-1}(b) = F_b \) is a submanifold (embedded)
- for \( x \in M \) \( F_x = \pi^{-1}(\pi(x)) \)

vertical space \((x \in M) : V_x = T_{\pi(x)}B \)

if \( M \) and \( B \) has a Riemannian metric then define horizontal space \( H_x = V_x^\perp \)

\[ d(\pi)_x : H_x \rightarrow T_{\pi(x)}B \]

**Defn:** \( \pi \) is a Riemannian submersion if this is an isometry.

\((M, <, >)\) Riemannian manifold

define a distance function \( d \):

\[ d(p, q) := \inf_{y \text{ from } p \text{ to } q} L(y) \]

- makes \( M \) into a metric space.

1. \( d(p, q) > 0 \)
2. \( d(p, q) = d(q, p) \)
1) $d(p,q) = 0 \implies p = q$. 

**Proof:** (sketch) 

$B_\varepsilon(p)$ normal ball s.t. $q \notin B_\varepsilon(p)$ 

then any curve from $p$ to $q$ has length $\geq \varepsilon$.

**Properties of Riemannian Submersions:**

1) $\pi$ decreases length: $d(x,y) \geq d(\pi(x), \pi(y))$ 

*Proof:* 

If $\gamma$ is a curve in $M$ then $L(\pi(\gamma)) \leq L(\gamma)$, since 

$$L(\gamma) = \int |\gamma'| \, dt = \int |\gamma_y + \gamma_h'| \, dt \geq \int |\gamma_h'| \, dt = \int |\pi(\gamma)'| \, dt$$

2) If $X$ is a v.f. on $B$ then there exists a unique v.f. $\overline{X}$ on $M$ s.t. $\overline{X}(y) \in T_y$ for all $y \in M$ and $d\pi(\overline{X}) = X$ 

($X$ is a horizontal lift) 

**Define:** 

$$\overline{X}(y) = (d\pi)^{-1}_{T_y}(X(\pi(y)))$$

$\overline{X}$ is smooth: locally, $\pi(x_1, \ldots, x_n + h) = (y_1, \ldots, y_n)$ 

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial y_i}, \quad \overline{X} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

Then $d\pi(\overline{X}) = X$ (but $\overline{X}$ need not be horizontal).
so, use Gram-Schmidt to project \( X(y) \) into \( H_y \).

\[
V_y = \left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^n \quad \text{call this projection } \overline{X}. \text{ Then } \overline{X} \text{ is smooth, horizontal.}
\]

**(3)** If \( y \) is a curve in \( B \) then \( \exists \text{ a local lift } \overline{\gamma} \text{ in } M \) s.t. \( \overline{X}(\overline{\gamma}) = \overline{y}, \overline{\gamma} \in H_{\overline{y}(t)} \)

\( \overline{\gamma} \) is called the horizontal lift of \( y \).

Assume \( y(t) \neq 0 \) \( \forall t \)

\( \text{Im}(y) \subset \mathcal{B} \text{ is a submanifold.} \)

Then \( L = \pi^{-1}(\text{Im}(y)) \text{ is a submanifold of } M \)

and on \( L \exists \text{ a } \nu \text{-f. } \overline{Z} \text{ which is a horizontal lift of } \nu \text{-f. } \overline{Y} \text{ on } L \)

\( \pi : L \to \text{Im}(y) \) is a Riemannian submersion.

Integral curves of \( \overline{Z} \) are the lifts \( \overline{Y} \) that we want to construct.

\( \overline{Z} \leftarrow \overline{Y}'' \) integral curves exist locally.

and \( \overline{Y} \) exists globally if \( M \) is metrically complete.

\((M, d) \]

\( \iff M \text{ is geodesically complete, i.e. geod. are defined for all } t \)

Hoff-Rinehart:
If \((M,d)\) metrically complete, lift \(\tilde{\gamma}\) is defined on \([0,t_0]\).

\(\tilde{\gamma}(t_i)\) is a Cauchy sequence: \(d(\tilde{\gamma}(t_i), \tilde{\gamma}(t_j)) \leq L(\tilde{\gamma})_{[t_i, t_j]}\)

\(\Rightarrow\) subseq \(\tilde{\gamma}(t_i) \to \tilde{p} \in \tilde{M}\), v.f. \(\tilde{Z}\) is defined at \(\tilde{p}\).

The lift \(\tilde{\gamma}\) is unique since I specify beginning point \(\tilde{\gamma}(0) \in \Gamma^{-1}(\gamma(0))\).

4) If \(\gamma\) is a geodesic in \(\tilde{M}\) \(\Rightarrow\) \(\tilde{\gamma}\) is a geodesic in \(M\).

\[L(\gamma) = L(\tilde{\gamma})\]

\(\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}\) and that \(\gamma\) is locally minimizing:

\[|\gamma'(t)| = |\tilde{\gamma}'(t)|\]

\(\Rightarrow\) \(\tilde{\gamma}\) is also locally minimizing since \(\tilde{\pi}\) decreases distance.

\(\Rightarrow\) \(\tilde{\gamma}\) is a geodesic.

\[|\tilde{\gamma}(t)| = |\gamma'(t)|\] is constant.

\[L(\pi(x)) \leq L(x)\]
5) If \( y \) is a geodesic in \( M \) s.t. \( y'(0) \in H_y(0) \) then 
\[ y'(t) \in H_y(t) \quad \forall t \quad \text{and} \quad \pi \circ y \text{ is a geodesic in } B. \]

Let \( \delta \) be the geodesic in \( B \) with 
\[ \delta(0) = \pi(y(0)) \quad \delta'(0) = \pi(y'(0)) \]

Let \( \hat{\delta} \) be the horizontal lift of \( \delta \), is a geodesic in \( M \)
(by part (4))
\[ \hat{\delta}(0) = y(0) \quad \hat{\delta}'(0) = y'(0) \]
\[ \Rightarrow y = \delta \quad \text{by uniqueness of geodesics} \]

Hence, if \( M \) is geodesically complete \( \Rightarrow \) \( B \) is geodesically complete.

6) Holonomy: Let \( y \in B \quad ( \gamma : [0,1] \rightarrow B ) \) be a path 
from \( q \) to \( q' \).

\[ \tau_y : F_y(0) \rightarrow F_y(1) \]
\[ x \rightarrow \gamma'(1) \text{ where } \gamma(0) = x \]

\[ (\tau_y)^{-1} = \tau_y^{-1} \]

(This is a diffeomorphism b/w the two fibers.)

7) If \( M \) is complete then \( \pi \) is a locally trivial fiber bundle with fiber \( F \)
\[ \pi^{-1}(U) = U \times F \]
Choose \( U \) to be a normal nbhd of \( p \).
let \( q \in U \) be unique minimizing geod. \( q_{pq} \) which defines a
diffeo \( U \to F_q \) and \( F_p \).

8) Fibers are constant distance apart.

\[
d(x, F_p) = \inf_{y \in F_p} d(x, y)
\]

\[
d(y, F_p) = d(y', F_q) \quad \forall y, y' \in F_p
\]

Exercise: Prove the converse. i.e. if condition (x) holds,

show that it's a Riemannian submersion.

Proof of (x):

Choose \( y \) a geod. in \( B \) from \( x \) to \( q \) s.t. \( L(y) = d(q, x) \).

Show: \( d(y, F_p) \leq d(y, F_q) \) (by symmetry they must be same).

Let \( \gamma \) be curve in \( M \) from \( y \) to \( F_p \)

\[
L(\gamma) = d(y, F_p)
\]

Then \( L(x \cdot \gamma) \leq L(\gamma) \)

\( \gamma \) is horizontal.

Then \( \gamma \) is horizontal.

\( L(x \cdot \gamma) \leq L(\gamma) \)

Examples:

1) \( B \times F \) product metric. \( B \times F \xrightarrow{\pi_1} B \)

2) warped product.

\( t \in C^0(B) \), \( \langle \cdot , \cdot \rangle_v \), \( \langle \cdot , \cdot \rangle_f \) define metric on \( B \times F \) by:
The fibers of $\mathcal{T}$ are totally geodesic. If the holonomy diffeomorphisms $\gamma_p: F_p \to F_q$, $\gamma(0) = p$, $\gamma(1) = q$ are isometric

\[\text{let } c: (-\varepsilon, \varepsilon) \to F_p \text{ be a curve with } v = c'(0) \in T F_p\]
\[\text{define variation } c(s, t) = \gamma(t)(c(s)) \text{ where } \gamma(t) = \gamma_{c(0), t}\]
\[\frac{\partial^2}{\partial t^2} < \frac{\partial c}{\partial s}, \frac{\partial c}{\partial s} > = 2 < \frac{\partial}{\partial t}, \frac{\partial c}{\partial s}, \frac{\partial c}{\partial s} >
\]
\[= 2 < \frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial c}{\partial s} > - 2 < \frac{\partial c}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial c}{\partial s} >
\]

since $\frac{\partial c}{\partial t}$ is normal to all fibers and $\frac{\partial c}{\partial s}$ tangent

\[= -2 < \frac{\partial c}{\partial s}, \frac{\partial c}{\partial s}, \frac{\partial c}{\partial s} > = -2 < \frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}, \frac{\partial c}{\partial s} >
\]

but $\frac{\partial c}{\partial s} = d(c_{x(1)})(\frac{\partial c}{\partial s}) = d(c_{x(t)}(v))$

Thus $\frac{\partial c}{\partial s}$ has constant length if $< \frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}, \frac{\partial c}{\partial s} > = 0$

if this is true for all $v$ (hence any $\frac{\partial c}{\partial s}$)

and for all $c(s)$ (hence any $\frac{\partial c}{\partial s}$) then $\beta = 0$

i.e. fibers are flat geodesic

If all fibers are totally geodesic, then they are all isometric to each other.
\[ \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_B + \langle f(p), \cdot \rangle \]

for \((f \times f) \in B \times F \quad B \times F \rightarrow B\quad \text{and} \quad T_B \perp T_F\]

\[ \int \frac{dt^2 + f^2(t)dt^2}{d\theta} \quad \text{polar coordinates} \]

\[ \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \quad d\theta \]

3) Let \( G \) be a Lie group acting on \((M, \langle \cdot, \cdot \rangle)\) by isometries and acting freely.

\( M_G \) is a manifold, \( M \rightarrow M_G = B \)

If a metric on \( B \) s.t. \( \pi \) is a Riemannian submersion.

\[ d\bar{\pi}|_H \pi_x \rightarrow T_{\pi(x)}B \quad \text{is an isom.} \]

Choose metric on \( T_{\pi(x)}B \) s.t. \( d\bar{\pi}|_H \pi_x \) is an isometry.

This metric on \( B \) is well-defined (independent of choice of \( x \)) at point in \( \pi^{-1}(\pi(x)) \):

say \( x, y \in \pi^{-1}(\pi(x)) \Rightarrow \exists g \in G \) s.t. \( y = gx \) and \( \& d\bar{\pi} \) takes vertical space to vertical space.

\( L_g \) preserves an orbit.

\[ dL_g \text{ takes } H_x \text{ form into } H_y \]

\[ dL_g | \pi(x) = \pi(y) \quad L_g \bar{\pi} = \bar{\pi} \]

\[ v \in H_x \]
e.g. $\mathbb{C}P^n$ complex lines in $\mathbb{C}^{n+1}$

$S^1 \subset S^{2n+1} \quad e^{i\theta} : (z_1, \ldots, z_{n+1}) \mapsto (e^{i\theta}z_1, \ldots, e^{i\theta}z_{n+1})$

this is an isometric action.

$S^{2n+1}/S^1 = \mathbb{C}P^n$

induces metric on $\mathbb{C}P^n$, called Fubini-Study metric.

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Riemannian submersions

Recall:

$\pi : (M, \langle, \rangle) \longrightarrow (B, \langle, \rangle)$

submersion $dt^2$ onto

$x \in M \quad V_x = T(\pi^{-1}(\pi(x)))$ vertical space

$H_x = V_x^\perp$ horizontal space

$\pi$ is called a Riemannian submersion if

$\frac{d}{dt}\pi(H_x) \rightarrow T\pi(x)B$

is an isometry.

Properties:

1) $d(\pi(x), \pi(y)) \leq d(x, y)$

2) $x$ v.f. on $B$, there exists horizontal lift $\tilde{x}$ v.f. on $M$

s.t. $\tilde{x}(y) = H_y$ and $d\pi(\tilde{x}) = x$

3) If $\gamma$ is a curve in $B$ (piecewise regular) then $\tilde{\gamma}$ horizontal
\[ \pi(t) = y \quad \text{and} \quad \pi'(t) \in H_y(t) \]

\( \pi \) is unique if we choose \( y(0) \in \pi^{-1}(y(0)) \)
(exists locally and \( t \) if \( M \) is complete)

4) If \( \gamma \) is a geodesic in \( B \) then \( \pi \) is a geod. in \( M \).

5) If \( c \) is a geod. in \( M \) s.t. \( c'(0) \in H_{\gamma(0)} \Rightarrow c(t) \in H_{\gamma(t)} \)
    \( \Rightarrow \pi(c) \) is a geodesic.

6) Two differ. If \( \gamma \) curve in \( B \) then
    \( \pi: \pi^{-1}(y(0)) \rightarrow \pi^{-1}(y(t)) \) differ.
    \( x \rightarrow \pi(x) \) where \( y(0) = x \).

7) \( \pi \) is locally trivial.

**O'Neill formula**

\[ \pi: M \rightarrow B \]

\[ \nabla^M \rightarrow \nabla^B \]

**Claim:**

\[ \left< \nabla^B_x Y, Z \right> = \left< \nabla^M_x Y, Z \right> \]

\( x, y, z \) v.f. on \( B \) and \( \tilde{x}, \tilde{y}, \tilde{z} \) their horizontal lifts.

**Proof:**

\[ \left< \nabla^B_x Y, Z \right> = \left< \tilde{x}, \tilde{y}, \tilde{z} \right> \]

Check it satisfies Levi-Civita properties.

\[ \text{a) product rule} \]

\[ \nabla_x (Y \cdot Z) = \nabla_x Y \cdot Z + Y \cdot \nabla_x Z \]

\[ \left< Y, Z \right> = \left< Y, Z \right> \]
Let \( \gamma(t) \) be an integral curve of \( X \)

\[
\langle y, z \rangle_{\gamma(t)} = \langle y, z \rangle_{\gamma(t)}
\]

**b) torsion-free**

\[
\langle [x, y], z \rangle = \langle \nabla^B_x y, z \rangle - \langle \nabla^B_x z, y \rangle
\]

\[
= \langle \nabla^M_x y, z \rangle - \langle \nabla^M_x z, y \rangle
\]

\[
= \langle [x, y], z \rangle = \langle [x, y], z \rangle
\]

\[
\tau_x ([x, y]) = [\tau_x x, \tau_x y]
\]

\[
= [x, y]
\]

\[
\tau_x (z) = z
\]

**hor+vert**

\[
\nabla^M_x y = \nabla^B_x y + A_x y
\]

**O'Neill tensor**

\[
A_x y = (\nabla^M_x y)_{\text{vert}} \quad \text{means project onto vertical space}
\]

\[
A_x y
\]
Lemma:  
(a) A is a tensor
(b) \( A : \mathcal{H} \times \mathcal{X} \rightarrow \mathcal{V} \) linear, skew (\( A_{x,y} = -A_{y,x} \))
(c) \( A_{x,y} = \frac{1}{2} \left[ \left[ x, y \right] \right] \)

Cor: \( A = 0 \) if and only if horizontal distribution is integrable.

Proof of Lemma:
(a) Linear over \( C^0 \), \( fX = f_0 X \)

(b) For linearity in \( X \) is clear

(c) For linearity in \( Y \):
\[
A_x(fY) = \left( \nabla^M_x (fY) \right)^\nu = \left( \nabla^M_x (f_0 Y) \right)^\nu
\]
\[
= \left( \left[ x(f_0), Y \right] + f_0 \nabla^M_x Y \right)^\nu
\]

This term goes away because we're taking vertical component
\[
\Rightarrow \quad f_0 A_x Y
\]

(b) \( A_{x,y} = -A_{y,x} \) if and only if \( A_x X = 0 \)
\[
\langle A_x X, U \rangle = \langle \nabla^M_x X, U \rangle
\]

(\( U \) is a v.f on \( M \) that is vertical \( U(x) \in \mathcal{V}_x \))
\[
= \overline{X \langle x, U \rangle} \quad - \langle x, \nabla^M_x U \rangle
\]
\[
= -\langle x, \nabla^M_x U \rangle = \frac{1}{2} U \left( \langle x, x \rangle \right) = 0
\]
\[ \langle x_2, x \rangle = \langle x, x \rangle \text{ is constant along a fiber} \]

**Proof:** It is equivalent to:

\[ \langle x, [x, u] \rangle = 0 \]

\[ [x, u] \in \mathfrak{v} ? \]

\[ \pi_* \left[ \dot{x} \right] = [\pi_* x, \pi_* u] = 0 \]

\[ \Rightarrow [x, u] \in \mathfrak{v} \]

\[ \left( \varepsilon \right) \left[ x, \bar{y} \right] = \nabla^M_x \bar{y} - \nabla^M_{\bar{y}} \bar{x} \]

\[ [x, \bar{y}]_{\pi_*} = A_{x \bar{y}} - A_{\bar{y} x} = 2 A_{x \bar{y}} \]
Riemannian submersion (formula for curvature using O'Neill tensor)

Notation: \( \pi: M \rightarrow B \) is a Riemannian submersion

\( \nabla^M \) = connection of \( M \)

\( \nabla^B \) = connection of \( B \)

\( R^M \) = curvature tensor of \( M \)

\( R^B \) = curvature tensor of \( B \)

for a vector field \( X \) on \( M \), \( X^h \) refers to its horizontal component and \( X^v \) its vertical component.

O'Neill tensor: let \( X, Y \) be vector fields on \( B \), \( \bar{X}, \bar{Y} \) their horizontal lifts (vector fields on \( M \)). Then we define a tensor \( A \) by:

\[
A_XY = (\nabla^H_X \bar{Y})^v
\]

Then the vector field curvature tensors of \( M \) and \( B \) are related by the following proposition:

Proof:

\[
\langle R^M(X, Y)Z, W \rangle = \langle R^B(X, Y)Z, W \rangle - 3 |A_XY|^2
\]

Proof: Step 1:

Let \( X, Y, Z, W \) be vector fields on \( B \) and \( \bar{X}, \bar{Y}, \bar{Z}, \bar{W} \) their horizontal lifts. Then:

\[
\langle \nabla^M_X \nabla^M_Y Z, W \rangle = \langle \nabla^B_X (\nabla^B_Y Z), W \rangle + \langle \nabla^M_X (A_Y Z), W \rangle
\]

\[
= \langle \nabla^B_X \nabla^B_Y Z + A_X \nabla^B_Y Z, W \rangle - \langle A_Y Z, \nabla^M_X W \rangle
\]

Since \( \langle A_Y Z, W \rangle = 0 \)

\[
= \langle \nabla^B_X \nabla^B_Y Z, W \rangle + \langle A_X \nabla^B_Y Z, W \rangle - \langle A_Y Z, \nabla^B_X W \rangle - \langle A_Y Z, A_X W \rangle
\]

(Inner product of vertical vectors with horizontal)
Step 2:
Now, curvature in M:
\[
\langle R^M(X, Y) Z, W \rangle := \langle \nabla^M_X \nabla^M_Y Z, W \rangle - \langle \nabla^M_Y \nabla^M_X Z, W \rangle - \langle \nabla^M [X, Y] Z, W \rangle
\]

By (1), we can write this as:
\[
= \langle \nabla^B_X \nabla^B_Y Z, W \rangle - \langle A_x Y, A_x X \rangle - \frac{\langle \nabla^B_X \nabla^B_Y Z, W \rangle + \langle A_x Y, A_x X \rangle}{0} - \frac{\langle \nabla^B [X, Y] Z, W \rangle}{0}
\]

= \langle \nabla^B_X \nabla^B_Y Z, W \rangle - \langle \nabla^B_X \nabla^B_Y Z, W \rangle - 1 A_x Y^2 - \langle \nabla^M [X, Y] Z, W \rangle
\]

Step 3:
\[
\langle \nabla^M [X, Y] Z, W \rangle = \langle \nabla^M [X, Y] \nabla^M Z, W \rangle + \langle \nabla^M [X, Y]^2 Z, W \rangle
\]

= \langle \nabla^B [X, Y] Z, W \rangle + \langle \nabla^M [X, Y]^2 Z, W \rangle
\]

= \langle \nabla^B [X, Y] Z, W \rangle + 2 \langle \nabla^B A_x Y Z, W \rangle
\]

Now, \( \nabla^M A_x y Y - \nabla^M A_x Y = [A_x Y, Y] \) = vertical vector field.

(\text{bracket of vertical & horizontal is vertical})

\[
\Rightarrow \langle \nabla^M A_x y Y, Z \rangle = \langle \nabla^M A_x Y, Z \rangle
\]
\[\begin{align*}
&= -\langle A_x y, \nabla_y^M x \rangle \\
&= -\langle A_x y, A_y x \rangle = |A_x y|^2
\end{align*}\]

Substituting this in (3), we get:

\[\langle \nabla^{[x, y]}_y x, x \rangle = \langle \nabla^{[x, y]}_x y, x \rangle + 2 |A_x y|^2\]  \hfill (4)

Substituting (4) in (\natural) (2) we get:

\[\langle R^M (x, y) y, x \rangle = \langle \nabla^B x \nabla^B y, x \rangle - \langle \nabla^B y \nabla^B x, x \rangle - |A_x y|^2 - \langle \nabla^B [x, y] y, x \rangle - 2 |A_x y|^2\]

\[= \langle R^B (x, y) y, x \rangle - 3 |A_x y|^2\]

So

\[\langle R^M (x, y) y, x \rangle = \langle R^B (x, y) y, x \rangle - 3 |A_x y|^2\]

\[\text{(Cor. \quad sec}_B (x, y) > \text{sec}_M (x, y)\]

(\text{Biemannian submersions increase curvature})
see notes on homepage. Fatness revisited.

\( \kappa: N \rightarrow \mathbb{B} \) is fat if \( \kappa^{-1}(b) \) are totally geodesic

and \( \sec (X,Y) = \frac{\langle V_x V_y X, X \rangle - \langle V_y V_x X, X \rangle}{\langle V_y V_x X, X \rangle} \)

\begin{align*}
&= \langle R^b(x,y) X, X \rangle + 2 \langle A_x Y, A_x X \rangle + 2 \left( \langle A_x X, A_x Y \rangle \right)

&= \langle R^b(x,y) Y, X \rangle + 3 \left| A_x Y \right|^2
\end{align*}

Example: \( \mathbb{C}P^n \) - Fubini-Study metric:

\( S^{2n+1}(1) \subset \mathbb{C}P^n \)

\[ \pi: S^{2n+1}(1) \rightarrow \mathbb{C}P^n \]

\[ \mathfrak{h}_\mathbb{C} = \{ \lambda \varphi \mid \lambda \in \mathbb{C} \} \]

action of \( S^1 = \{ e^{i\theta} \mid \theta \in \mathbb{C} \} \) on \( S^{2n+1} \)

\[ S^1 \times S^{2n+1} \rightarrow S^{2n+1} \]

\[ (e^{i\theta}, x) \rightarrow e^{i\theta} x \]

\[ \forall \varphi, \psi \in S^{2n+1}(1) \quad \kappa(\varphi) = \kappa(\psi) \Rightarrow \varphi = e^{i\theta} \psi \quad \text{for some } e^{i\theta} \in S^1 \]

\[ S^{2n+1}(1)/S^1 = \mathbb{C}P^n \]

action of \( S^1 \) is isometric \( \Rightarrow \kappa \) is a Riemannian submersion

\[ x \in S^{2n+1}(1) \quad V_x \text{ spanned by } \left. \frac{d}{d\theta}\right|_{\theta=0} (e^{i\theta} x) = iX \]

\[ H_x = \left\{ x, x^* \right\}^{-1} \text{ inside } \mathbb{R}^{2n+2} \]

\[ iX = V_x \]
\[ T[\mathbf{x}] \mathbb{C}P^n = \left[ (x, \nu) \right] (x, \nu) \sim (e^{i\theta}x, e^{i\theta} \nu) \]

\( \nu \in H_x \), \( \nu \perp x_x \), \( \nu \perp ix \).

great circle in \( \mathbb{C}P^n \)

\[ \gamma_\nu(t) \quad |\nu| = 1 \]

\[ \gamma_\nu(0) = x, \quad \nu \perp x, \quad ix \]

\[ \gamma_\nu(t) = \left[ \cos t + \sin t \nu \right] \quad \text{equiv. class of great circle} \]

all geodesics are closed of length \( \pi \).

\[ \gamma_\nu(\pi) = [-x] = [x] \]

and they don't close earlier.

Proof:

\[ \gamma_\nu(t) = \gamma_\nu(0) = [x] = [\cos t x + \sin t \nu] \quad t < \pi \]

\( \Rightarrow e^{i\theta}x = (\cos t)x + (\sin t)\nu \) for some \( \theta \).

\( 0 = \langle e^{i\theta}x, \nu \rangle = \sin t \langle \nu, \nu \rangle \neq 0. \)

Exercise: \( \mathbb{C}P^1 = S^2(\frac{i}{2}) \)

What is \( A_x Y ? \)

Lemma: \( [x] \in \mathbb{C}P^n \), \( \nu, \omega \in H_x \), \( \nu \perp x, i x \)

\[ A_x \omega = \langle i \nu, \omega \rangle i x \]
Con: Assume \( p \neq e \), o.n.

\[
\begin{align*}
\text{if } \omega \perp i \nu & \implies \sec (\omega, \nu) = 1 \\
\text{if } \omega = +i \nu & \implies \sec (\nu, \omega) = 4
\end{align*}
\]

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Recall: \( \pi: M^{n+k} \rightarrow B^n \) Riemannian submersion

\[
\nu_e = T_{\pi^{-1} (e)}
\]

\[
H_x = 2_x \quad T_\pi: H_x \rightarrow T_{\pi(x)} B \quad \text{is a geodesic on } B
\]

\( \gamma \) a geod. on \( M \) s.t. \( \gamma'(0) \in H_{\gamma(0)} \Rightarrow \gamma'(t) \in H_{\gamma(t)} \)

and \( \pi \circ \gamma \) is a geod. in \( B \).

O'Neil tensor: \( A: H \times H \rightarrow V \)

\[
A_x \nu = \left( \nabla^M_x \nu \right) = \frac{1}{2} [\nu, \nu]
\]

O'Neil formula:

\[
\sec_B (x, \nu) = \frac{\sec_M (x, \nu) + \|A_{x, \nu}\|^2}{\sec_M (x, \nu)}
\]

\( \mathbb{CP}^m \) Fubini-Study metric

General fact: If \( G \) acts isometrically and freely on \( M \), then

\( \mathfrak{g} \) metric on \( B = M/G \) s.t. \( \chi: M \rightarrow M/G \) is a R.S.
$\mathbb{C}P^n \subset S^{2n+1}$, $S^{2n+1} / S^1 = \mathbb{C}P^n$

$S^1 \times S^{2n+1} \stackrel{(e^{i\theta}, \mathbf{v})}{\longrightarrow} S^{2n+1}$, $e^{i\theta} \mathbf{v}$ parallelogramm action.

$\pi: S^{2n+1} \to \mathbb{C}P^n$ gives a metric on $\mathbb{C}P^n$.

$\mathbf{x} \in S^{2n-1}, \mathbf{[x]} \in \mathbb{C}P^n$

$T_{[x]} \mathbb{C}P^n = \left\{ (x, \mathbf{v}) \mid \mathbf{v} \perp x, ix \right\}$

$H_x = \left\{ \mathbf{v} \mid \mathbf{v} \perp ix, ix \right\}$

$(x, \mathbf{v}) \sim (e^{i\theta} x, e^{i\theta} \mathbf{v})$

$\gamma(t) = x \cos t + \mathbf{v} \sin t \in S^{2n+1}$ (geodesic)

$\gamma'(t) \in \mathbb{H} \gamma(t)$

$\gamma'(t) \perp \gamma(t)$ and $i, \gamma(t)$

$\langle -x \sin t + \mathbf{v} \cos t, x \cos t + \mathbf{v} \sin t \rangle = 0$

$\langle -x \sin t + \mathbf{v} \cos t, ix \cos t + iv \sin t \rangle = \langle x, ix \rangle \sin t \cos t$

$\langle \mathbf{v}, iv \rangle \sin t \cos t$

$= 0$
Lemma: \( v, w \in H_x \), \( A_v w = \langle x, iw \rangle i x \) \( \text{(O'Neill computations for \( CP^n \))} \)

Proof: \( A_v w = (\frac{2 ^ {n+1}}{\langle w, w \rangle}) v \)

\[ y(t) = x \cos t + v \sin t \quad \Rightarrow \quad \dot{y}(0) = v \]

\( R^{2n+2} \ni w(t) = w \cos t + z \sin t \quad \text{some } z \)

\[ w(t) \in H_y(t) \]

\[ [w] \text{ is a vector field along } [y(t)] \]

one condition:

\[ 0 = \langle w(t), \dot{y}(t) \rangle = \langle w \cos t + z \sin t, x \cos t + v \sin t \rangle \]

\[ = \langle w, v \rangle \cos t \cos t + \langle z, x \rangle \sin t \cos t \]

\[ + \langle z, w \rangle \sin ^2 t \]

need

\[ \langle z, v \rangle = 0 \]

\[ \langle x, w \rangle = -\langle w, v \rangle \]

another condition:

\[ 0 = \langle w(t), i \dot{y}(t) \rangle = \langle w \cos t + z \sin t, ix \cos t + iv \sin t \rangle \]

\[ = \langle w, iv \rangle \cos t \sin t + \langle z, ix \rangle \sin t \cos t \]

\[ + \langle z, iv \rangle \sin ^2 t \]

get additional conditions:

\[ \langle z, iv \rangle = 0 \]

\[ \langle z, ix \rangle = -\langle w, iv \rangle \]
So, \( V \overrightarrow{\omega} = \left( \nabla \overrightarrow{\omega} \right) = \overrightarrow{\omega} (\overrightarrow{\omega}(e)) \)

\[ = \overrightarrow{\omega}(\overrightarrow{z}) = \langle \overrightarrow{z}, i \overrightarrow{x} \rangle i \overrightarrow{x} = -\langle \overrightarrow{\omega}, i \overrightarrow{\omega} \rangle i \overrightarrow{x} \]

\[ = -\langle i \overrightarrow{\omega}, i \overrightarrow{\omega} \rangle i \overrightarrow{x} = \langle \overrightarrow{\omega}, i \overrightarrow{\omega} \rangle i \overrightarrow{x} \]

\[ \square \]

Recall, \( \sec_B(\overrightarrow{v}, \overrightarrow{w}) = \sec_M(\overrightarrow{v}, \overrightarrow{w}) + 3\|A_{\overrightarrow{v}}\overrightarrow{w}\|^2 \) if \( \overrightarrow{v}, \overrightarrow{w} \) o.n.

So we get a corollary of the above lemma:

\[ \cos \left( \sec_{\text{proj}}(\overrightarrow{v}, \overrightarrow{w}) = \sec_{\text{geom}}(\overrightarrow{v}, \overrightarrow{w}) + 3\langle \overrightarrow{v}, \overrightarrow{w} \rangle^2 \right)^2 \]

\[ = 1 + 3\langle \overrightarrow{v}, i \overrightarrow{w} \rangle^2 \]

So, \( 1 \leq \sec_{\text{proj}} \leq 4 \), and:

\[ \sec(\overrightarrow{v}, \overrightarrow{w}) = 1 \text{ iff } \overrightarrow{v} \perp i \overrightarrow{w} \text{ "real plane"} \]

\[ \sec(\overrightarrow{v}, \overrightarrow{w}) = 4 \text{ iff } \overrightarrow{v} = \pm i \overrightarrow{w} \text{ "complex plane"} \]

Exercise: totally geodesic submanifolds of \( \mathbb{C}P^n \)

\[ V = T_{[x]} \mathbb{C}P^n \quad V \text{ is called complex if } iV = V \]

\[ V \text{ is called totally real if } V \perp iV \]

\[ \bullet \text{ } V \text{ is complex } \Rightarrow \exp_{[x]}(V) = \mathbb{C}P^k < \mathbb{C}P^n \text{ tot. geod. } \]

metric on \( \mathbb{C}P^k \text{ Fubini-Study} \)
* $V$ is totally real $\Rightarrow \exp \left[ \frac{1}{2} \partial^2 \right] \in \mathbb{R} P^n \subset \mathbb{C} P^n$ real pt.s.

* complex structure on $M$:

$$J : T_p M \rightarrow T_p M \text{ linear and } J^2 = -\text{Id}$$

$J$ is integrable if $J$ is multiplication by $i$

complex mfd $M$ locally $\mathbb{C}$, interchanges hol.

Theorem. $J$ is integrable if

$$N_J(x, y) = [x, y] + J([x, Jy] + J[x, Jy] - [x, Jy]) = 0$$

Defn. $(M, J, \langle \cdot, \cdot \rangle)$, $\langle \cdot, \cdot \rangle$ is Hermitian if $\langle Jv, Jw \rangle = \langle v, w \rangle$.

$\Rightarrow J$ is skew-symmetric since $J^2 = -\text{Id}$

$\langle Jv, w \rangle = \langle v, Jw \rangle$ $\Rightarrow Jv \perp v$.

Defn. $(M, J, \langle \cdot, \cdot \rangle)$ is called Kähler if Hermitian and $\nabla J = 0$.

$(\Rightarrow$ if $V$ is parallel along $Y$ then $JV$ is also parallel $)$

Exericse

$$\nabla_X J(Y) = \nabla_X (JY) - J(\nabla_X Y)$$

* $J$ is almost complex if $\nabla J = 0 \Rightarrow J$ is complex

Defn. Kähler form: $\omega \in \wedge^2 \Rightarrow \omega \langle v, w \rangle = \langle Jv, w \rangle = -\langle v, Jw \rangle$

$(\omega \text{ is a 2-form})$

$$\nabla J = 0 \iff d\omega = 0$$
Exercise: $\mathbb{C}P^n$ with Fubini-Study metric is Kähler ($\nabla J = 0$).

Defn. holomorphic curvature: $\sec (\nu, \nu) = \sec \Theta$.

e.g. $\mathbb{C}P^n, \sec \Theta = 1$ (and conversely).

($\mathbb{C}P^n$ is characterized by this property)

Defn. bisectional curvature: $\nu_{\nu}$.

\[ \sec (\nu_\nu) + \sec (\nu_\nu, J\nu) = \langle R(X, JX)Y, JY \rangle \]

Theorem. Frankel conjecture (Shi-Yau)
If $M$ is compact, $\chi = 0$, Kähler + positive holomorphic bisectional curvature $\Rightarrow$ biholomorphic to $\mathbb{C}P^n$.

Hurewicz-Rips

Theorem. (a) $M$ is metrically complete (for distance $d$) iff it is geodesically complete.

(b) If complete, then any 2 points can be connected by a minimal geodesic: $L(y) = d(y(0), y(t))$.

(c) $\exp_p (B_r(0)) = B_r(p)$

(d) If $\exp_p$ is defined on $T_pM$ for some $p$, then also for all $p$. 