Riemannian Four-Manifolds with Nonnegative Curvature and Continuous Symmetry

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1 Introduction

A longstanding problem in Riemannian geometry is to determine which compact manifolds $M^n$ admit Riemannian metrics $g$ with positive sectional curvature. When $n = 2$ one can apply the Gauss-Bonnet theorem to conclude that a compact surface admitting a metric with positive curvature must be diffeomorphic to $S^2$ or $RP^2$. Compact positively curved manifolds (henceforth CPCM's) of dimension three were classified in [4]. They are diffeomorphic to spherical space forms. Four dimensional CPCM's are poorly understood. Synge's theorem [2] tells us that $\pi_1(M^4)$ is trivial or $\cong Z_2$ depending on whether $M^4$ is orientable or not, while in [6] it is shown that the sum of the Betti numbers of $M^4$ is less than a universal (astronomical) constant $C_4$. On the other hand, the only four dimensional CPCM's known are diffeomorphic to $S^4$, $RP^4$, or $CP^2$.

One of the goals of this thesis is to study a special class of four dimensional CPCM's: those which admit a continuous family of symmetries. A topological classification is given by

Theorem 1.0.1. Let $(M^4, g)$ be a connected orientable CPCM. Suppose $(M^4, g)$ admits a nontrivial Killing vector field (i.e. an isometric $R$-action). Then $M^4$ is homeomorphic to $S^4$, or $CP^2$.

The reasoning used in the proof of this theorem can be sharpened to prove
Theorem 1.0.2 Let \((M^4, g)\) be a one-connected compact nonnegatively curved Riemannian manifold. If \((M^4, g)\) admits a nontrivial Killing vector field then \(M^4\) is homeomorphic to \(S^4\), \(CP^2\), \(S^3 \times S^2\), or \(CP^2\# \pm CP^3\).

As in theorem 1.0.1, theorem 1.0.2 gives a complete classification since each of the manifolds listed in the conclusion of theorem 1.0.2 supports a metric \(g\) satisfying the hypotheses of the theorem (see [1]).

A compact nonnegatively curved manifold homeomorphic to \(S^2 \times S^2\) or \(CP^2\# \pm CP^2\) which admits a nontrivial Killing vector field cannot have strictly positive curvature by theorem 1.0.1, so it is natural to investigate the distribution of flat (i.e. zero curvature) two-planes in such a Riemannian manifold. Some partial results in this direction can be found in sections 3.2 and 3.3.

In section 4 we consider compact nonnegatively curved Riemannian manifolds diffeomorphic to \(S^2 \times S^2\) and \(CP^2\# \pm CP^2\) which admit nontrivial isometric actions of \(SU(2)\). This symmetry assumption leads to a very strong structure theorems – essentially classification up to isometry – in most cases. Nonetheless, Kuranishi discovered some very interesting examples of nonnegatively curved metrics on \(S^3 \times S^2\) which admit isometric \(SO(3)\) actions.

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2 Preliminaries

This section gathers the geometric tools that will be used in the sequel. It may be skipped on a first reading without loss of continuity and used as a reference when needed.

In this section, \((M^n, g)\) will be a complete connected Riemannian manifold with nonnegative curvature.

2.1 The Geometry of Manifolds with Nonnegative Curvature

Theorem 2.1.1 (Frankel) Let \(N_1^{k_1}, N_2^{k_2} \subseteq M^n\) be closed totally geodesic submanifolds with \(k_1 + k_2 \geq n\), at least one of which is compact. Then either \(N_1^{k_1} \cap N_2^{k_2} \neq \emptyset\) or there is a minimizing geodesic segment \(\eta : [0, 1] \to M^n\) with

1. \(\eta(0) = p_1 \in N_1^{k_1}\) and \(\eta(1) = p_2 \in N_2^{k_2}\)
2. \(g(\eta'(0), T_{\eta(0)}N_1^{k_1}) = g(\eta'(1), T_{\eta(1)}N_2^{k_2}) = 0\)
3. There is a parallel normal vector field \(X : [0, 1] \to TM^n\) along \(\eta\) such that the sectional curvature is zero for each of the two-planes \(\text{span}(\eta'(t), X(t)) \subseteq T_{\eta(t)}M^n\)

Proof: Suppose \(N_1^{k_1} \cap N_2^{k_2} = \emptyset\). Then there is a shortest geodesic segment \(\eta : [0, 1] \to M^n\) from \(N_1^{k_1}\) to \(N_2^{k_2}\). The first variation formula for arclength implies that \(g(\eta'(0), T_{\eta(0)}N_1^{k_1}) = g(\eta'(1), T_{\eta(1)}N_2^{k_2}) = 0\), for otherwise one could shorten \(\eta\) while keeping its endpoints in \(N_1^{k_1}\) and \(N_2^{k_2}\).

Let \(X : [0, 1] \to TM^n\) be a parallel normal vector field along \(\eta\) such that \(X(0) \in T_{\eta(0)}N_1^{k_1}\) and \(X(1) \in T_{\eta(1)}N_2^{k_2}\). Such a \(X\) exists; letting \(V\) be the vector space of parallel normal vector fields along \(\eta\), \(V_1 = \{X \in V | X(0) \in T_{\eta(0)}N_1^{k_1}\}\) and \(V_2 = \{X \in V | X(1) \in T_{\eta(1)}N_2^{k_2}\}\), then \(\dim(V_1 \cap V_2) \geq \dim V_1 + \dim V_2 - \dim V = k_1 + k_2 - (n-1) \geq 1\).

Consider the variation \(\eta_s\) of \(\eta\) given by \(\eta_s(t) = \exp(sX(t)), s \in R, t \in [0, 1]\). Then

\[
\frac{d^2}{ds^2}(\text{length}(\eta_s))|_{s=0} = \int_0^1 g(R(X(t), \eta'(t))X(t), \eta'(t))dt \leq 0
\]
where \( R(\cdot, \cdot) \) is the curvature tensor of \( M^n \). Since \( \eta \) is a shortest path from \( N_1^{k_i} \) to \( N_2^{k_i} \), we conclude that

\[
\frac{d^2}{ds^2} (\text{length}(\eta_s))|_{s=0} = 0
\]

forcing \( g(R(X(t), \eta'(t)), X(t), \eta'(t)) = 0 \) for all \( t \in [0, l] \). \( \Box \)

In the second case of theorem 2.1.1 one can prove more with the following lemma. It will be used in sections 3.2, 3.3, and 4.

**Lemma 2.1.2** Let \( \gamma_i : [0, S] \rightarrow M^n, i = 1, 2 \) and \( \eta_0 : [0, l] \rightarrow M^n \) be geodesic segments. Suppose \( \eta_0(0) = \gamma_1(0), \eta_0(l) = \gamma_2(0), g(\eta_0'(0), \gamma_1'(0)) = g(\gamma_1'(0), \gamma_2'(0)) = 0 \), and \( X : [0, l] \rightarrow TM^n \) is a parallel normal vector field along \( \eta_0 \) with \( X(0) = \gamma_1'(0), X(l) = \gamma_2'(0) \). Define \( \eta : [0, S] \times [0, l] \rightarrow M^n \) by \( \eta(s, t) = \exp(sX(t)) \) and \( \eta_s : [0, l] \rightarrow M^n \) by \( \eta_s = \eta \circ (s, \text{id}) \). Then either

1. \( \text{length}(\eta_s) = \text{length}(\eta_0) \) for every \( s \in [0, S] \).
2. \( \eta \) is an isometric immersion.
3. \( \eta_s (\frac{\partial}{\partial t}) \) is parallel along each \( \eta|_{[0, S] \times \{t\}} \), where \( \frac{\partial}{\partial t} \) is the second coordinate vector field on \( [0, S] \times [0, l] \).

**Proof:** Assume that \( \text{length}(\eta_s) \geq \text{length}(\eta_0) \) for all \( s \in [0, S] \). Let \( [0, S_1] \subseteq [0, S] \) be the largest subinterval of \( [0, S] \) for which

1. \( \text{length}(\eta_s) = \text{length}(\eta_0), s \in [0, S_1] \)
2. \( \eta|[0, S_1] \times [0, l] \) is an isometric immersion,
3. \( \eta_s (\frac{\partial}{\partial t}) \) is parallel along \( \eta|[0, S_1] \times \{t\} \) for all \( t \in [0, l] \).

We will assume that \( S_1 < S \), and prove the lemma by contradiction. Consider the Jacobi field \( J_t = \eta_s (\frac{\partial}{\partial t}) \) along the geodesic \( \eta|[0, S_1] \times \{t\} \). This is a parallel Jacobi field on the segment \( \eta|[0, S_1] \times \{t\} \), so \( (\nabla_{\frac{\partial}{\partial s}} J_t)(S_1, t) = 0 \). Hence by theorem 1.29 \([2]\), \( |J_t(s)| \leq |J_t(S_1)| \) for \( s \in [S_1, S_1 + \varepsilon] \), with equality holding for every \( s \in [S_1, S_1 + \varepsilon] \) only when \( J_t \) is parallel on the interval \([S_1, S_1 + \varepsilon]\).
and \( \epsilon > 0 \) depends only on curvature bounds for \( M^n \) in a neighborhood of \( \eta([0,S] \times [0,l]) \). For \( s \in [S_1, S_1 + \epsilon] \),

\[
\text{length}(\eta_s) = \int_0^l |\eta'_s(t)| dt = \int_0^l |J_t(s)| dt \\
\leq \int_0^l |J_t(S_1)| dt = \text{length}(\eta_{S_1}) = \text{length}(\eta_0)
\]

Since \( \text{length}(\eta_s) \geq \text{length}(\eta_0) \) we conclude that \( J_t \) is parallel on \([0,S_1 + \epsilon]\) for every \( t \in [0,l] \). This shows that \( \eta|_{[0,S_1 + \epsilon] \times [0,l]} \) is an isometric immersion and \( \eta_s(\frac{\partial}{\partial t}) \) is parallel along \( \eta|_{[0,S_1] \times [0,l]} \) for each \( t \in [0,l] \). This contradicts the definition of \( S_1 \in (0,S) \), proving lemma 2.1.2. \( \square \)

**Corollary 2.1.3** If \( \gamma_t, \eta_0, \eta_t \), etc. are as in lemma 2.1.2 and \( \text{dist}(\gamma_t, \gamma_2) = \text{length}(\eta_0) \), then \( \eta \) is a totally geodesic isometric immersion.

**Proof:** By lemma 2.1.2, \( \eta \) is an isometric immersion and \( \eta_t(\frac{\partial}{\partial s}) \) is parallel along \( \eta|_{[0,S_1] \times [0,l]} \) for each \( t \in [0,l] \), so if \( II \) is the second fundamental form of the immersion \( \eta \), then

\[
II(\eta_s(\frac{\partial}{\partial s}), \eta_s(\frac{\partial}{\partial t})) = II(\eta_s(\frac{\partial}{\partial s}), \eta_s(\frac{\partial}{\partial t})) = 0
\]

The curves \( \eta_s \) are all minimizing paths from \( \gamma_t \) to \( \gamma_2 \), so \( II(\eta_s(\frac{\partial}{\partial s}), \eta_s(\frac{\partial}{\partial t})) = 0 \) as well. Hence \( II \equiv 0 \), and \( \eta \) is totally geodesic. \( \square \)

We now turn to the geometry of geodesic triangles in nonnegatively curved manifolds.

**Definition 2.1.4** If \( p, q \in M^n \), and \( \text{dist}(p, q) = l \), let

\[
\text{Min}_p(q) = \{ \gamma : [0,l] \rightarrow M^n | \gamma(0) = p, \gamma(l) = q, \text{and} |\gamma'| \equiv 1 \}
\]

= \{ minimizing paths from p to q \}

\[
\text{Dir}_p(q) = \{ \text{initial directions of } \gamma \in \text{Min}_p(q) \}
\]

= \{ \gamma'(0) | \gamma \in \text{Min}_p(q) \}

\[
L_p(q, r) = L_p(\text{Dir}_p(q), \text{Dir}_p(r))
\]
Theorem 2.1.5 (Toponogov) 1. If \( p_1, p_2, p_3 \in M^n \), then \( L_{p_1}(p_2, p_3) + L_{p_2}(p_1, p_3) + L_{p_3}(p_1, p_2) \geq \pi \).

2. If equality holds in 1, and \( \gamma_2 \in \text{Min}_{p_2}(p_2), \gamma_3 \in \text{Min}_{p_3}(p_3) \) with \( L(\gamma_2'(0), \gamma_3'(0)) = L_{p_3}(p_2, p_1) \), then there is an embedded, flat, totally geodesic triangular surface \( \Delta^2 \subseteq M^n \) with geodesic edges \( \gamma_2, \gamma_3, \) and \( \eta \in \text{Min}_{p_2}(p_2) \).

Proof: Let \( l_{ij} = \text{dist}(p_i, p_j), 1 \leq i, j \leq 3, i \neq j \), and construct a triangle \( \Delta \subseteq E^2 \) with vertices \( \tilde{p}_i \in E^2 \) and \( \text{dist}(\tilde{p}_i, \tilde{p}_j) = l_{ij} \).

Given \( \gamma_2 \in \text{Min}_{p_2}(p_2), \gamma_3 \in \text{Min}_{p_3}(p_3) \) we may apply Toponogov's theorem [2] to conclude that \( L_{p_1}(\gamma_2'(0), \gamma_3'(0)) \geq L(\overline{p_1 p_2}, \overline{p_1 p_3}) \). Hence \( L_{p_1}(p_2, p_3) = L(\text{Dir}_{p_1}(p_2), \text{Dir}_{p_1}(p_3)) \geq L(\overline{p_1 p_2}, \overline{p_1 p_3}) \). Cyclic repetition of this reasoning gives

\[
L_{p_1}(p_2, p_3) + L_{p_2}(p_1, p_3) + L_{p_3}(p_1, p_2) \\
\geq L(\overline{p_1 p_2}, \overline{p_1 p_3}) + L(\overline{p_2 p_1}, \overline{p_2 p_3}) + L(\overline{p_3 p_1}, \overline{p_3 p_2}) \\
= \pi
\]

The second part of the theorem follows from the discussion of the equality case of Toponogov's theorem in [2]. \( \Box \)

2.2 The Geometry of Nonnegatively Curved Manifolds with Symmetry

In this section we fix a compact Lie group \( G \) with Lie algebra \( L(G) \) and an isometric action \( \Phi : G \times M^n \rightarrow M^n \). \( G(p) \), for \( p \in M^n \) will denote the \( G \) orbit \( \{ g \cdot p | g \in G \} \) of \( p \), and \( G_p \) will denote the isotropy subgroup \( \{ g \in G | g \cdot p = p \} \). The notation for the induced action of \( G \) on \( TM^n \) will be similar, except that elements of \( TM^n \) will be used in place of elements of \( M^n \).

Definition 2.2.1 A geodesic segment \( \gamma : [0, l] \rightarrow M^n \) is a \( G \)-geodesic segment if \( g(\gamma'(t), T_{\gamma(t)}G(\gamma(t))) = 0 \) for all \( t \in [0, l] \), and \( G \)-minimizing if \( \text{length}(\gamma) = \text{dist}(G(\gamma(0)), G(\gamma(l))) \).

Lemma 2.2.2 1. (Clairaut) If \( \gamma : [0, l] \rightarrow M^n \) is a geodesic with \( g(\gamma'(0), T_{\gamma(0)}G(\gamma(0))) = 0 \) then \( \gamma \) is a \( G \)-geodesic segment.

2. A \( G \)-minimizing segment is a \( G \)-geodesic segment.
3. If $\gamma : [0, l] \rightarrow M^n$ is $G$-minimizing, then $G_{\gamma(t)}$ is constant on $(0, l)$, and $G_{\gamma(t)} \equiv G_{\gamma}$, for $t \in (0, l)$.

Proof: 1 and 2 follow immediately from the first variation formula for arclength.

To see 3, pick $t \in (0, l)$, and suppose $G_{\gamma(t)} \neq G_{\gamma}$. Then there exists $g \in G_{\gamma(t)}$ such that $g \cdot \gamma'(t) \neq \gamma'(t)$, for otherwise $G_{\gamma(t)} \subseteq G_{\gamma}$. The segments $\gamma\big|_{[0,t]}$ and $g \cdot \gamma\big|_{[t,l]}$ fit together to give a nonsmooth path $\eta$ from $\gamma(0) \in G(\gamma(0))$ to $g \cdot \gamma(l) \in G(\gamma(l))$ and $\text{length}(\eta) = \text{length}(\gamma)$. This contradicts the minimality of $\gamma$. □

When $G$ acts freely on $M^n$ or more generally when the $G$ action has only on orbit type, then $M^n/G$ inherits a canonical smooth quotient Riemannian metric, the submersion metric. The submersion formula of O'Neill guarantees that this metric has nonnegative curvature, so one may apply the results in section 2.1. Proposition 2.2.7 below and the discussion preceding it provide a partial analog for the second variation formula in $M^n/G$ when the $G$ action has more than one orbit type, and the quotient metric isn’t smooth.

The first step is the formulation of $G$-parallelism given in definition 2.2.5.

Definition 2.2.3 If $\eta : [0, l] \rightarrow M^n$ is a $G$-geodesic, then the vertical bundle $Vert^G_{\eta}$ along $\eta$ has as fiber over $\eta(t)$,

$$Vert^G_{\eta}(t) = \text{span}(\{T_{\eta(t)}G(\eta(t)))\} \left\{ \frac{d}{d\epsilon}(\exp_{\eta(t)}X \cdot \eta'(t)\bigg|_{\eta(t)} \bigg| X \in L(G_{\eta(t)}) \right\}$$

where $L(G_{\eta(t)})$ is the Lie algebra of $G_{\eta(t)}$.

If $G(\eta(t))$ is a principal orbit then $Vert^G_{\eta}$ will have fiber $T_{\eta(t)}G(\eta(t))$ over $\eta(t)$ since $G_{\eta(t)}$ acts trivially on $\nu_{\eta(t)}G(\eta(t))$. Hence $Vert^G_{\eta}$ will actually be a smooth vector bundle near $\eta(t)$. Alternatively, note that the quotient $M^n \rightarrow M^n/G$ defines a Riemannian submersion near $\eta(t)$ and the vertical bundle is just the restriction of the usual vertical bundle to $\eta$.

Lemma 2.2.4 $Vert^G_{\eta}$ is a smooth vector bundle along $\eta$.

Proof: Fix $t_0 \in [0, l]$, and consider $Vert^G_{\eta}$ near $\eta(t_0)$. 

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First suppose $\eta(t_0) \in \text{Fix}(G,M^n) = \{p \in M^n | g \cdot p = p, \ \forall g \in G\}$. Then the $G$ action is locally smoothly equivalent via the exponential map $T_{\eta(t_0)}M^n \to M^n$ to the orthogonal action of $G$ on $T_{\eta(t_0)}M^n$. The exponential local equivalence carries a piece of $\eta$ to a line segment in $T_{\eta(t_0)}M^n$, so the smoothness of $\text{Vert}^G_\eta$ near $\eta(t_0)$ is immediate.

In the general case, let $H = G_{\eta(t_0)}$, and let $W \subseteq L(G)$ be a complement to $L(H)$ in $L(G)$. Consider the bundles $\text{Vert}^H_\eta$ and $\tilde{W}$ near $\eta(t_0)$, where the fiber of $\tilde{W}$ at $\eta(t)$ is $\{|\frac{d}{ds}(\exp sX) \cdot \eta(t)|_{s=0} | X \in W\}$. Both bundles are smooth, and their fibers are linearly independent at $\eta(t_0)$ since $G_{\eta(t_0)}$ preserves the splitting $T_{\eta(t_0)}M^n = T_{\eta(t_0)}G(\eta(t_0)) \oplus \nu_{\eta(t_0)}G(\eta(t_0))$.

**Claim:** $\text{Vert}^G_\eta = \text{Vert}^H_\eta \oplus \tilde{W}$ near $t_0$.

**Proof of claim:** The fiber of $\text{Vert}^G_\eta$ at $\eta(t)$ is spanned by $T_{\eta(t)}G(\eta(t))$ and $\{|\frac{d}{ds}(\exp sX) \cdot \eta(t)|_{s=0} | X \in L(G_{\eta(t)}(G))\}$. Assuming $\eta(t)$ is within a normal tubular neighborhood of $G(\eta(t_0))$, then $G(t) \subseteq G_{\eta(t_0)}$. In fact, if $t \neq t_0$ then $G(\eta(t)) = \{g \in G_{\eta(t_0)} | g \cdot \eta(t_0) = \eta(t_0)\} = H_{\eta(t_0)}$. Therefore if $t \neq t_0$ and $\eta(t)$ is close to $\eta(t_0)$, then $\{|\frac{d}{ds}(\exp sX) \cdot \eta(t)|_{s=0} | X \in L(G_{\eta(t)})\} = \{0\}$, and $\text{Vert}^G_\eta(t) = T_{\eta(t)}G(\eta(t))$. Since $(\text{Vert}^H_\eta \oplus \tilde{W})(t)$ spans $T_{\eta(t)}G(\eta(t))$ for $\eta(t)$ close to $\eta(t_0)$, the proof is complete. \[\square\]

**Definition 2.2.5** If $\eta : [0, l] \to M^n$ is a $G$-geodesic segment and $X : [0, l] \to TM^n$ is a vector field along $\eta$, then $X$ is $G$-parallel if $X$ is a parallel section of the smooth bundle $\text{Hor}^G_\eta = (\text{Vert}^G_\eta)^\perp$ along $\eta$. In other words $g(X(t), \text{Vert}^G_\eta(t)) = 0$ and $\nabla_{\eta(t)}X(t) \in \text{Vert}^G_\eta(t)$ for every $t \in [0, l]$.

**Remark 2.2.6** Since $\text{Vert}^G_\eta$ is smooth, one can extend any $X(t_0) \in \text{Hor}^G_\eta(t_0) = (\text{Vert}^G_\eta(t_0))^\perp$ to a $G$-parallel section in the usual way. This is the motivation for lemma 2.2.4.

**Proposition 2.2.7** Let $\gamma_1, \gamma_2 : [0, S] \to M^n$, $\eta_0 : [0, l] \to M^n$ be $G$-geodesic segments with $\eta_0([0, l]) \subseteq (M^n)^o = \text{the union of the principal } G \text{-orbits}$. Suppose $\eta_0(0) = \gamma_1(0)$, $\eta_0(l) = \gamma_2(0)$, $g(\eta_0'(0), \gamma_1'(0)) = g(\eta_0'(l), \gamma_2'(0)) = 0$, and $X : [0, l] \to TM^n$ is a $G$-parallel vector field along $\eta_0$ with $X(0) = \gamma_1'(0)$, $X(l) = \gamma_2'(0)$. Then either $\text{dist}(G(\gamma_1(s)), G(\gamma_2(s))) < \text{length}(\eta_0) = l$ for some $s \in [0, S]$, or $X$ is parallel along $\eta_0$ in the ordinary sense of the word.
Corollary 2.2.8 Let $\gamma_1$, $\gamma_2$, $\eta_0$, and $X$ be as in 2.2.7, and suppose $\text{dist}(G(\gamma_1(s)), G(\gamma_2(s))) \geq \text{length}(\eta_0) - 1$ for every $s \in [0, S]$. Define $\eta : [0, S] \times [0, l] \to M^n$ by $\eta(s, t) = \exp(sX(t))$. Then $\eta$ is a totally geodesic isometric immersion of the Euclidean rectangle $[0, S] \times [0, l]$. Moreover $\eta$ is orthogonal to the $G$-action: $g(\eta_s(Y), T_{\eta(s,t)}G(\eta(s,t))) = 0$ for every $Y \in T_{\eta(s,t)}([0, S] \times [0, l])$ and every $(s, t) \in [0, S] \times [0, l]$.

Proof of corollary 2.2.8: By proposition 2.2.7, $X$ is a parallel vector field along $\eta$. Now lemma 2.1.2 and corollary 2.1.3 can be applied, showing that $\eta$ is a totally geodesic isometric immersion. To see that $\eta$ is orthogonal to the $G$-action, note that each segment $\eta_s = \eta \circ (s, id_{[0,l]} : [0, l] \to M^n$ is $G$-minimizing, so $g(\eta_s'(t), T_{\eta(s,t)}G(\eta_s(t))) = 0$ for all $t \in [0, l]$, while the segments $\eta \circ (id_{[0,S]}, t) : [0, S] \to M^n$ are $G$-geodesic segments by part 1 of lemma 2.2.2. □

Proof of 2.2.7: The idea is to try to apply the ordinary second variation formula in the orbit space $M^n/G$, which has nonnegative curvature in its smooth part by O'Neill's curvature formula for Riemannian submersions[2].

Let $\eta$ and $\eta_s$ be defined as in lemma 2.1.2. Picking $\delta \in (0, 1/2)$, $s \in [0, S]$, we have

$$\text{dist}(G(\gamma_1(s)), G(\gamma_2(s))) = \text{dist}(G(\eta(s,0)), G(\eta(s,l)))$$

$$\leq \text{dist}(G(\eta(s,0)), G(\eta(s,\delta))) + \text{dist}(G(\eta(s,\delta)), G(\eta(s,l-\delta))) + \text{dist}(G(\eta(s,l-\delta)), G(\eta(s,l)))$$

$$\leq \text{dist}(G(\eta(s,\delta)), G(\eta(s,l-\delta))) + \text{dist}(\eta(s,0), \eta(s,\delta)) + \text{dist}(\eta(s,l-\delta), \eta(s,l))$$

Looking at the last two terms we have

$$\text{dist}(\eta(s,0), \eta(s,\delta)) \leq \text{length}(\eta_s|_{[0,\delta]}$$

$$\leq \text{length}(\eta_0|_{[0,\delta]}) + \delta C_1 s^2$$

where $C_1$ is a constant independent of $\delta$ and $s$. Likewise

$$\text{dist}(\eta(s,l-\delta), \eta(s,l))$$

$$\leq \text{length}(\eta_0|_{[l-\delta,l]}) + \delta C_2 s^2$$

for some $C_2$ independent of $\delta$ and $s$. 
The first term can be estimated by working in the smooth, nonnegatively curved Riemannian manifold \((M^n)^\circ / G\). Define \(\eta_s : [\delta, l - \delta] \to M^n / G\) by
\[
\eta_s = \pi \circ \eta_s \big|_{\delta, l - \delta}
\]
where \(\pi : M^n \to M^n / G\) is the quotient map. For small \(s\), \(\eta_s\) defines a a variation of \(\eta_0\) in \((M^n)^\circ / G\) = the smooth part of \(M^n / G\), with parallel variation vector field \(\bar{X} = \pi_* \circ X\). Hence the second variation formula gives
\[
\frac{d^2}{ds^2} \text{length}(\eta_s) \bigg|_{s=0} = \int_{0}^{l - \delta} g(\bar{R}(\eta'_0(t), \bar{X}(t)) \eta_0'(t), \bar{X}(t)) dt
\]

\[
= I_s
\]
where \(\bar{R}\) is the curvature tensor of \((M^n)^\circ / G\). Combining these estimates we get
\[
dist(G(\gamma_1(s)), G(\gamma_2(s))) \\
\leq \text{length}(\eta_0) + (\delta(C_1 + C_2) + I_s + \text{error}(s)) s^2
\]
where \(\lim_{s \to 0}(\text{error}(s)) = 0\).

Since \(I_s\) is monotone in \(\delta\), either \(\text{dist}(G(\gamma_1(s)), G(\gamma_2(s))) < \text{length}(\eta_0) = l\) for some \(s \in [0, S]\) or \(I_s \equiv 0\). In the latter case \(g(\bar{R}(\eta'_0(t), \bar{X}(t)) \eta_0'(t), \bar{X}(t)) = 0\) for every \(t \in [0, l]\), implying via O'Neill's formula that the horizontal lift of \(\bar{X}\), i.e. \(X\), is parallel along \(\eta_0\).

2.3 A Lemma

Lemma 2.3.1 Let \(\rho : G \to O(n)\) be an orthogonal representation of a compact Lie group \(G\), and suppose \(v, w \in E^n \setminus \{0\}\). If \(\rho\) has no trivial summands then \(\sup_{g \in G} \langle g \cdot v, w \rangle \geq 0\) with equality only if \(\rho\) is reducible and \(G(v), G(w)\) lie in mutually orthogonal direct summands of \(\rho\).

Proof: If \(x \in E^n\), then \(\text{Average}_{g \in G}(g \cdot x) \in \text{Fix}(G, E^n) = \{0\}\), so \(\text{Average}_{g \in G}(g \cdot x) = 0\). Therefore \(\sup_{g \in G} \langle g \cdot v, w \rangle \geq \text{Average}_{g \in G} \langle g \cdot v, w \rangle = \langle \text{Average}_{g \in G} (g \cdot v, w) \rangle = \langle 0, w \rangle = 0\). Equality holds only if \(\langle g \cdot v, w \rangle = 0\) for every \(g \in G\), which gives \((\text{span}(G(v)), \text{span}(G(w))) = \{0\}\).
3 The Proofs of Theorems 1.0.1 and 1.0.2

The main goal in this section is to prove theorems 1.0.1 and 1.0.2. Section 3.1 reduces these two theorems to the analysis of three cases, which are dealt with in sections 3.2, 3.3, and 3.4. In addition to pieces of the proof of 1.0.1 and 1.0.2, sections 3.2 and 3.3 contain some further information on the structure of certain manifolds satisfying the hypotheses of 1.0.2. See theorem 3.2.2 and theorem 3.3.6.

3.1 Reduction of 1.0.1 and 1.0.2

Theorems 1.0.1 and 1.0.2 are corollaries of

Theorem 3.1.1 Let \((M^4, g)\) be a four dimensional, connected, compact, nonnegatively curved manifold admitting a nontrivial Killing vector field. Then

1. \(\chi(M^4) \leq 4\)

2. \(\chi(M^4) = 4\) only if \((M^4, g)\) has some flat two-planes, i.e. two-planes with zero sectional curvature.

Proof of theorems 1.0.1 and 1.0.2 using theorem 3.1.1:

The hypotheses of theorem 1.0.1 together with Synge's theorem [2] imply that \(\pi_1(M^4) = \{1\}\). Hence \(b_1(M^4) = rank(H_1(M^4, Z)) = 0\) and by Poincaré duality \(b_3(M^4) = 0\). Now

\[
\chi(M^4) = \sum_{i=0}^{4} (-1)^i b_i(M^4)
\]
\[
= b_0(M^4) + b_2(M^4) + b_4(M^4)
\]
\[
= 2 + b_2(M^4)
\]

Applying theorem 3.1.1, we get \(b_2(M^4) = \chi(M^4) - 2 \leq 1\). Freedman's topological classification of simply connected four-manifolds [3] now gives \(M^4 \approx S^4, CP^2\).

The proof of theorem 1.0.2 from theorem 3.1.1 is similar except that one has to check that all the possibilities for an intersection form for \(M^4\) are realized by those of the listed manifolds when \(rank(H_2(M^4, Z)) \leq 2\). □
Proof of theorem 3.1.1: To prove theorem 3.1.1 we will assume that \( \chi(M^4) \geq 4 \) and will show that \( \chi(M^4) = 4 \) and the sectional curvature of \((M^4, g)\) isn't strictly positive.

The hypotheses imply that the isometry group of \((M^4, g)\) is a positive dimensional compact Lie group \(G\). We pick a circle subgroup \(S^1 \subseteq G\) and we will use the corresponding effective isometric \(S^1\) action \(S^1 \times M^4 \rightarrow M^4\).

Let \(\text{Fix}(S^1, M^4) = \{ p \in M^4 | S^1(p) = p \}\) be the fixed point set of \(S^1\) in \(M^4\). Smith theory and theorem [5] imply that

1. \(\chi(\text{Fix}(S^1, M^4)) = \chi(M^4)\).

2. \(\text{Fix}(S^1, M^4)\) is a disjoint union of (a finite number of) totally geodesic submanifolds, each of which has even codimension in \(M^4\).

In our case, \(\text{Fix}(S^1, M^4)\) is a disjoint union of isolated points and totally geodesic surfaces with \(\chi(\text{Fix}(S^1, M^4)) = \chi(M^4) \geq 4\). This implies that \(\text{Fix}(S^1, M^4)\) contains one of the following subsets:

1. Two disjoint totally geodesic surfaces.

2. A totally geodesic surface and two isolated points.

3. Four isolated points.

In sections 3.2, 3.3, and 3.4 respectively we show that cases 1, 2, 3 each lead to the conclusion that \(\chi(M^4) = \chi(\text{Fix}(S^1, M^4)) = 4\) and the sectional curvature of \(M^4\) isn't strictly positive. This proves theorem 3.1.1. \(\square\)

3.2 \(\text{Fix}(S^1, M^4)\) Contains Two Totally Geodesic Surfaces

In this section we study the geometry of a four dimensional nonnegatively curved manifold \((M^4, g)\) and an effective isometric \(S^1\) action \(S^1 \times M^4 \rightarrow M^4\) whose fixed point set \(\text{Fix}(S^1, M^4)\) contains a pair of disjoint totally geodesic surfaces, \(N_1^2\) and \(N_2^2\). In corollary 3.2.3 we show that \(\chi(M^4) = \chi(\text{Fix}(S^1, M^4)) \leq 4\), and in observation 3.2.1 we point out that the sectional curvature of \((M^4, g)\) isn't strictly positive.

Observation 3.2.1 Theorem 2.1.1 implies that \(M^4\) doesn't have strictly positive sectional curvature.
The main goal of this section is to prove

**Theorem 3.2.2.**

1. \( \text{Fix}(S^1, M^4) = N_1^2 \sqcup N_2^2 \).
2. \( S^1 \) acts freely on \( M^4 \setminus (N_1^2 \sqcup N_2^2) \).
3. \( (M^4 \setminus (N_1^2 \sqcup N_2^2))/S^1 \) is isometric to \( N_1^2 \times (0, l) \), where \( l = \text{dist}(N_1^2, N_2^2) \).

For the proof of theorem 3.2.2 see lemmas 3.2.5 and 3.2.7.

Part 3 of theorem 3.2.2 is really a splitting theorem for \( M^4/S^1 \). We avoid this formulation since it would involve the unnecessary complication of working in the (a priori) nonsmooth orbit space \( M^4/S^1 \).

**Corollary 3.2.3.** \( \chi(M^4) = \chi(\text{Fix}(S^1, M^4)) \leq 4 \)

**Proof:** By theorem 3.2.2 we have \( \chi(\text{Fix}(S^1, M^4)) = \chi(N_1^2) + \chi(N_2^2) \leq 2 + 2 = 4 \).

**Lemma 3.2.4.** Suppose \( p \in N_1^2 \) satisfies \( \text{dist}(p, N_2^2) = \text{dist}(N_1^2, N_2^2) = l \), and \( \eta_0 : [0, l] \rightarrow M^4 \) is a geodesic segment with \( \eta_0(0) = p \), \( \eta_0(l) = q \in N_2^2 \), and \( \text{length}(\eta_0) = l \). Then

1. Any \( X(0) \in T_pN_1^2 \) extends to an \( S^1 \)-parallel (see definitions 2.2.1 and 2.2.3), parallel, Jacobi field \( X : [0, l] \rightarrow TM^4 \) along \( \eta_0 \) corresponding to a variation through geodesic segments normal to \( N_1^2 \).
2. \( \text{Dist}(\exp(sX(0)), N_2^2) = l \) for every \( s \geq 0 \).
3. \( \text{Dist}(p', N_2^2) = l \) for every \( p' \in N_1^2 \).

**Proof:** We will assume the reader is familiar with section 2.2. Since \( \text{length}(\eta_0) = \text{dist}(N_1^2, N_2^2) = \text{dist}(p, q) = \text{dist}(G(p), G(q)) \), we have \( g(\eta_0'(0), T_pN_1^2) = g(\eta_0'(l), T_qN_2^2) = 0 \) and \( \eta_0 \) is an \( S^1 \)-minimizing geodesic segment (see definition 2.2.1). Lemma 2.2.2 implies that \( S^1_{\eta_0}(t) \) is trivial for \( t \in (0, l) \). The representation of \( S^1 \) on \( T_pM^4 \) preserves \( T_pN_1^2 \), and so \( S^1(\eta_0'(0)) \subseteq \nu_pN_1^2 \) is the normal space of \( N_1^2 \) at \( p \). Therefore \( X(0) \in T_pN_1^2 \) extends to an \( S^1 \)-parallel vector field \( X : [0, l] \rightarrow TM^4 \) along \( \eta_0 \) by remark 2.2.6. Note that since \( X(l) \in (\text{Vert}_{\eta_0'}(l))^2 \) and \( g(X(l), \eta_0'(l)) = 0 \), we have \( X(l) \in T_{\eta_0(l)}N_2^2 \).

Define \( \gamma_1 : [0, \infty) \rightarrow M^4 \), \( \gamma_2 : [0, \infty) \rightarrow M^4 \) by \( \gamma_1(s) = \exp(sX(0)) \) and \( \gamma_2 = \exp(sX(l)) \). Then \( \gamma_1([0, \infty)) \subseteq N_1^2 \) and \( \gamma_2([0, \infty)) \subseteq N_2^2 \) since \( N_1^2 \)
and $N_2^2$ are totally geodesic. Applying corollary 2.2.8, we get parts 1 and 2 of the lemma. To get part 3 of the lemma, pick $X(0) \in T_pN_1^2$ so that $\exp(sX(0)) = p'$ for some $s \in [0, \infty)$ and then use part 2. \(\square\)

**Lemma 3.2.5**

1. $\exp\big|_{D_{\nu}N_1^2} : D_{\nu}N_1^2 \to M^4$ is a diffeomorphism onto $M^4 \setminus N_2^2$

2. $S^1$ acts freely on $M^4 \setminus (N_1^2 \cup N_2^2)$

**Proof:** It follows from the previous lemma that if $p \in N_1^2$ is arbitrary and $\eta_0 : [0, l] \to M^4$ is a geodesic segment with $\eta_0'(0) \in \nu N_1^2$, then $\eta_0$ is an $S^1$-minimizing geodesic segment from $\eta_0(0) \in N_1^2$ to $\eta_0(l) \in N_2^2$, and in fact a minimizing segment from $\eta_0(0)$ to $N_2^2$ or from $\eta_0(l)$ to $N_2^2$. Therefore the normal injectivity radius of $N_1^2$ is $l$, $\exp\big|_{D_{\nu}N_1^2}$ is a diffeomorphism onto its image, and $S^1$ acts freely on $\exp(D_{\nu}N_1^2) \setminus N_2^2$.

Any geodesic extension of $\eta_0$ will fail to be minimizing by part 3 of lemma 2.2.2, so $\text{dist}(q, N_1^2) \leq l$ for every $q \in M^4$ and $\text{dist}(q, M^4) = l$ only when $q \in N_1^2$. This proves that $\exp(D_{\nu}N_1^2) = M^4 \setminus N_2^2$ and part 2 of the lemma. \(\square\)

Let $\pi : D_{\nu}N_1^2 \to N_1^2$ be the bundle projection, and define $P : M^4 \setminus N_2^2 \to N_1^2$ by $P = \pi \circ (\exp\big|_{D_{\nu}N_1^2})^{-1}$.

**Lemma 3.2.6** $P$ is an $S^1$-equivariant Riemannian submersion.

**Proof:** The $S^1$ equivariance of $P$ follows from that of $\pi$ and $\exp\big|_{D_{\nu}N_1^2}$.

The fiber $P^{-1}(p)$, for $p \in N_1^2$, is just $\exp(D_{\nu}pN_1^2)$. If $q \in P^{-1}(p) \setminus \{p\}$, and $q = \eta_0(t)$ for some $t \in (0, l)$, and some geodesic segment $\eta_0 : [0, l] \to M^4$ with $\eta_0'(0) \in \nu N_1^2$, then $T_q(P^{-1}(p))$ is spanned by $\eta_0'(t)$ and $T_qS^1(q)$. Hence $\nu_qP^{-1}(p)$ is spanned by the various $X(t) \in T_qM^4$ as $X$ runs over all the Jacobi vector fields produced by lemma 3.2.4. Since $P_*X(t) = X(0)$ for such a Jacobi field, $\nu_qP^{-1}(p) : \nu_qP^{-1}(p) \to T_pN_1^2$ is an orthogonal linear transformation. Therefore $P$ is a Riemannian submersion. \(\square\)

Define $d : M^4 \setminus (N_1^2 \cup N_2^2) \to (0, l)$ by $d(q) = \text{dist}(q, N_2^2)$. This is a smooth Riemannian submersion since the normal injectivity radius of $N_1^2$ is $l$.
Lemma 3.2.7 1. The map $P \times d : M^4 \setminus (N_1^2 \amalg N_2^2) \rightarrow N_1^2 \times (0, l)$ is a Riemannian submersion if $N_1^2 \times (0, l)$ is given the product metric.

2. $P \times d$ induces an isometry $\overline{P \times d} : (M^4 \setminus (N_1^2 \amalg N_2^2))/S^1 \rightarrow N_1^2 \times (0, l)$.

Proof: To see 1, pick $q \in M^4 \setminus (N_1^2 \amalg N_2^2)$ and note that the horizontal subspaces of the Riemannian submersions $P$ and $d$, i.e. $(ker P_q)^H$ and $(ker d_q)^H$, respectively, are orthogonal subspaces of $T_q M^4$.

2 follows immediately from the $S^1$-equivariance (or $S^1$-invariance) of $P \times d$. □

3.3 $Fix(S^1, M^4)$ Contains a Totally Geodesic Surface and Two Isolated Points

In this section we consider a four-dimensional, compact, connected, nonnegatively curved Riemannian manifold $(M^4, g)$ and an isometric $S^1$-action on $(M^4, g)$ whose fixed point set contains at least two isolated points, $p_1$, $p_2$, and a totally geodesic surface $N^2$. We will show that the sectional curvature of $(M^4, g)$ isn't strictly positive and that $Fix(S^1, M^4) = \{p_1, p_2\} \cup N^2$, which implies that $\chi(M^4) \leq 4$. In addition, we show that a dense open subset of $M^4$ admits an $S^1$-equivariant Riemannian submersion to an open rectangle $(0, l_1) \times (0, l) \subseteq E^2$, where the rectangle is given the trivial $S^1$-action. This can be interpreted as saying that an open dense subset of the quotient $M^4/S^1$ admits a Riemannian submersion to $(0, l_1) \times (0, l)$. The geometric idea here is to apply the argument of 2.1.1 and 2.1.2 in the orbit space $M^4/S^1$. The "totally geodesic" submanifolds will be $N^2$ and a geodesic segment $\gamma$ which runs from $p_1$ to $p_2$. It turns out that the endpoints of $\gamma$ don't cause problems in the argument; in fact, they play a crucial role. As in section 3.2, the proof presented below will avoid explicit mention of the orbit space $M^4/S^1$.

Let $\gamma : [0, l_1] \rightarrow M^4$ be a minimizing geodesic segment with $\gamma(0) = p_1$, and $\gamma(l_1) = p_2$.

Lemma 3.3.1 At least one of the two endpoints $p_1, p_2$ of $\gamma$ satisfies $\text{dist}(p_i, N^2) = \text{dist}(\text{Im} \gamma, N^2) = \text{dist}(S^1(\text{Im} \gamma), N^2) = l$.

Proof: Pick $p \in \text{Im}(\gamma)$ closest to $N^2$, i.e. $\text{dist}(p, N^2) = \text{dist}(\text{Im} \gamma, N^2) = \text{dist}(S^1(\text{Im} \gamma), N^2)$. If $p = p_1$ or $p_2$ there is nothing to prove, so assume $p \in$
interior(Im\(\gamma\)), and let \(\eta_0 : [0, l] \rightarrow M^4\) be a geodesic from \(p\) to \(q \in N^2\) with \(\text{length}(\eta_0) = l = \text{dist}(p, N^2)\). The first variation formula shows that \(\eta_0\) hits \(\text{Im}(\gamma)\) and \(N^2\) orthogonally, since \(\text{length}(\eta_0) = \text{dist}(p, N^2) = \text{dist}(\text{Im}(\gamma), N^2)\).

\(\eta_0\) is an \(S^1\)-geodesic segment since \(\text{length}(\eta_0) = \text{dist}(S^1(p), N^2)\). If \(X(0) \in T_p\text{Im}(\gamma)\), then by remark 2.2.6, \(X(0)\) can be extended to an \(S^1\)-parallel normal vector field \(X : [0, l] \rightarrow TM^4\) along \(\eta_0\), and it is easy to see that \(X(l) \in T_qN^2\) since \(X(q) \in (\text{Vert}_{\eta_0}(l))^{\perp}\) (see definition 2.2.5).

Define \(\gamma_1 : [0, S] \rightarrow M^n\) and \(\gamma_2 : [0, S] \rightarrow M^n\) by \(\gamma_1(s) = \exp(sX(0))\) and \(\gamma_2(s) = \exp(sX(l))\) respectively, where \(S = \sup\{s \in R | \exp(sX(s)) \in \text{Im}(\gamma)\}\). We can apply corollary 2.2.8 to conclude that \(\text{dist}(\gamma_1(S), N^2) = l\).

This establishes the lemma since \(\gamma_1(S)\) is an endpoint of \(\text{Im}(\gamma)\).

**Lemma 3.3.2** Assume without loss of generality that \(\text{dist}(p_1, N^2) = \text{dist}(\text{Im}(\gamma), N^2) = l\). Then

1. \(\text{dist}(\cdot, N^2)\) is constant along \(\text{Im}(\gamma)\). In particular, \(\text{dist}(p_2, N^2) = \text{dist}(\text{Im}(\gamma), N^2)\).

2. Let \(N_1^3 = S^1(\text{Im}(\gamma))\). Then \(N_1^3\) is a smooth surface.

3. Given \(p \in N_1^3\) and a geodesic segment \(\eta : [0, l] \rightarrow M^4\) with \(\eta'(0) \in \nu_pN_1^3\), we have \(\eta(l) \in N^2\). Moreover, if \(X(0) \in T_pN_1^3\) satisfies \(g(X(0), T_pS^1(p)) = 0\), then \(X(0)\) extends to an \(S^1\)-parallel, parallel Jacobi vector field along \(\eta\) corresponding to a variation of \(\eta\) through geodesic segments normal to \(N_1^3\).

**Proof:** Once again, the idea is to exploit corollary 2.2.8.

Let \(\eta_0 : [0, l] \rightarrow M^4\) be a geodesic segment with \(\eta_0(0) = p_1, \eta_0(l) \in N^2,\) and \(\text{length}(\eta_0) = \text{dist}(p_1, N^2) = \text{dist}(\text{Im}(\gamma), N^2) = l\). By the first variation formula \(\sup_{h \in S^1} \{g(h \cdot \eta_0(0), \gamma'(0))\} \leq 0\), while by lemma 2.3.1 \(\sup_{h \in S^1} \{g(h \cdot \eta_0(0), \gamma'(0))\} \geq 0\). Hence \(g(h \cdot \eta_0(0), \gamma'(0)) = g(\eta_0(0), h \cdot \gamma'(0)) = 0\) for every \(h \in S^1\). Fix \(h \in S^1\) for a moment, and extend \(X(0) = \gamma'(0)\) to an \(S^1\)-parallel vector field \(X : [0, l] \rightarrow TM^4\) along the \(S^1\)-minimizing geodesic segment \(h \cdot \eta_0\). We have \(X(l) \in T_{\eta_0(l)}N^2\) because \(g(X(l), h \cdot \eta_0(l)) = g(X(l), \text{Vert}_{\eta_0}(l)) = 0\) and \(\text{span}(h \cdot \eta_0(l), \text{Vert}_{\eta_0}(l)) = \nu_{h \cdot \eta_0(l)}N^2\). Let \(S = l_1 = \text{dist}(p_1, p_2), \gamma_1 = \gamma : [0, S] \rightarrow M^4, \gamma_2 : [0, S] \rightarrow M^4, \gamma_2(s) = \exp(sX(l)) \in N^2\). Since \(S^1(\eta_0(t)) = S^1(\eta_0(t)) = \{e\}\) for \(t \in (0, l)\) by part 3 of lemma 2.2.2, corollary 2.2.8 may be applied to show that \(\text{dist}(\gamma_1(s), N^2) = l\) for every \(s \in [0, l]\), and
to get a totally geodesic isometric immersion $\eta_h : [0, l_1] \times [0, l] \to M^4$ which is orthogonal to the $S^1$ action. For every $s \in [0, l_1]$, $\eta_h \circ (s, id_{[0, l]}) : [0, l] \to M^4$ is an $S^1$-minimizing geodesic segment from $\eta_h(s, 0) = \gamma(s)$ to $N^2$, and the variation of $\eta_h \circ (s, id_{[0, l]})$ obtained by varying $s$ gives an $S^1$-parallel, parallel Jacobi vector field along $\eta_h \circ (s, id_{[0, l]})$ with initial condition a multiple of $\gamma'(s)$.

Consider $N^2 = S^1(\gamma) \subseteq M^4$. The minimizing segments $h \cdot \gamma$, $h \in S^1$, have initial velocities in the two dimensional space $V = \{(h \cdot \gamma(0))| h \in S^1\} \subseteq T_p M^4$. Let $D_{i_1} V = \{v \in V| |v| < l_1\}$, $D_{i_2} V = \{v \in V| |v| \leq l_1\}$. Then $N^2 = \exp(D_{i_1} V)$, and $N^2 \setminus \{p_2\} = \exp(D_{i_2} V)$ a smooth surface. Analogous reasoning shows that $N^2 \setminus \{p_1\}$ is a smooth surface as well. This proves 2.

To prove 3, we may assume that $p = \gamma(s)$ for some $s \in [0, l_1]$. The initial velocities of the minimizing segments $\eta_h \circ (s, id_{[0, l]})$ constructed above fill up $\nu_{\mu} N^2$ as $h$ runs over $S^1$ since they lie in a two dimensional subspace of $T_p M^4$ which is orthogonal to $\gamma'(s)$ and $T_p S^1(p)$. Any $X(0) \in T_{p} N^2$ satisfying $g(X(0), T_p S^1(p)) = 0$ must be a multiple of $\gamma'(s)$, so the desired extension $X: [0, l] \to TM^4$ of $X(0)$ was constructed above by varying $\eta_h \circ (s, id_{[0, l]})$.

Lemma 3.3.3 \hspace{1cm} 1. $\exp\big|_{D_{i_2} N^2} : D_{i_1} \nu N^2 \to M^4 \setminus N^2$ is a diffeomorphism onto $M^4 \setminus N^2$.

2. $S^1$ acts freely on $M^4 \setminus (N^2 \sqcup N^2)$.

Proof: Proceed as in the proof of lemma 3.2.5, except replace the use of lemma 3.2.4 with lemma 3.3.2. \qed

Corollary 3.3.4 $Fix(S^1, M^4) = \{p_1, p_2\} \sqcup N^2$.

Proof: By lemma 3.3.2, $Fix(S^1, M^4) = Fix(S^1, N^2 \sqcup N^2) = \{p_1, p_2\} \sqcup N^2$. \qed

By lemma 3.3.3 we may define $P = \pi \circ (\exp\big|_{D_{i_2} N^2})^{-1} : M^4 \setminus N^2 \to N^2$ where $\pi : D_{i_2} \nu N^2 \to N^2$ is the bundle projection. Let $\tilde{N}^2 = N^2 \setminus \{p_1, p_2\}$ and let $\tilde{\pi} : \tilde{N}^2 \to \tilde{N}^2 \sqcup S^1$ be the quotient map.

Lemma 3.3.5 $\tilde{P} = \tilde{\pi} \circ (P^{-1}|_{\tilde{N}^2}) : \tilde{P}^{-1}(\tilde{N}) \to \tilde{N}^2 \sqcup S^1$ is an $S^1$-equivariant Riemannian submersion if $\tilde{N}^2 \sqcup S^1$ is given the submersion metric induced by $\tilde{\pi} : \tilde{N}^2 \to \tilde{N}^2 \sqcup S^1$. 

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Proof: The $S^1$-equivariance of $\tilde{P}$ follows from that of $P$ and $\tilde{\pi}$. Pick $q \in P^{-1}(\tilde{N}^2)$. There is a unique $p \in \tilde{N}^2$ and a unique geodesic segment $\eta : [0,1] \to M^4$ such that $\eta'(0) \in \nu_p\tilde{N}^2$ and $q = \eta(t)$ for some $t \in [0,1]$. Let $\eta$ be a nonzero $S^1$-parallel, parallel Jacobi vector field along $\eta$ given part 3 of lemma 3.3.2. Since $|\tilde{P}_\ast X(t)| = |\tilde{\pi}_\ast P_\ast X(t)| = |\tilde{\pi}_\ast X(0)| = |\pi(0)| = |X(t)|$, it remains only to show that $X(t)$ is orthogonal to the fiber $\tilde{P}^{-1}(\tilde{P}(q)) = \exp(D_{t\nu}\tilde{N}^2)_{\tilde{\pi}(\nu)}$ along $\eta$.

Let $\eta_\lambda : [0,1] \to M^4$ be a smooth family of geodesic segments with $\eta_\lambda(0) \in \nu_p\tilde{N}^2$, $\eta_0 = \eta$. Assume that $Y = \frac{d}{dt}\eta_\lambda|_{t=0}$ is a nontrivial Jacobi field along $\eta_0 = \eta$. Then the vertical space of $\tilde{P}$ at $q$, i.e. $T_q\tilde{P}^{-1}(\tilde{P}(q))$, is spanned by $Y(t)$, $\eta'(t)$, and $T_qS^1(q)$. We already know that $g(X(t),\eta'(t)) = g(X(t),T_qS^1(q)) = 0$, so it suffices to check that $g(X(t),Y(t)) = 0$. Applying the Jacobi equation along $\eta$, we have

$$
\left(\frac{d^2}{dt^2}g(X,Y)\right)(t) = g(\nabla^2 g(X,Y),X)
=g(R(\eta'(t),Y(t))\eta'(t),X(t)) = g(R(\eta'(t),X(t))\eta'(t),Y(t))
=g(\nabla^2 g(X,Y),t\eta(t)) = 0.
$$

Since $g(Y(0),X(0)) = \frac{d}{dt}(g(X(t),Y(t)))(0) = 0$, we have $g(Y(t),X(t)) = 0$ for every $t \in [0,1]$. \qed

Let $d : M^4 \setminus (N^2 \Pi N^2) \to (0,1)$ be given by $d(q) = \text{dist}(q,N^2)$. It follows from lemma 3.3.3 that $d$ is a smooth Riemannian submersion. Let $\tilde{M}^4 = P^{-1}(\tilde{N}^2) \setminus \tilde{N}^2$, and let $\Phi = \tilde{P} \big|_{\tilde{M}^4} \times d \big|_{\tilde{M}^4} : \tilde{M}^4 \to \tilde{N}^2 \times S^1 \times (0,1)$.

Theorem 3.3.6 $\Phi$ is an $S^1$–equivariant Riemannian submersion.

Proof: $\Phi$ is $S^1$–equivariant since $\tilde{P}$ and $d$ are $S^1$–equivariant. To see that $\Phi$ is a Riemannian submersion, note that the horizontal distributions of the Riemannian submersions $\tilde{P} \big|_{\tilde{M}^4}$ and $d \big|_{\tilde{M}^4}$ are mutually orthogonal. \qed

3.4 $\text{Fix}(S^1,M^4)$ Contains Four Isolated Points

In this section we study the geometry of a four-dimensional, connected, compact, nonnegatively curved Riemannian manifold $(M^4,g)$ with an isometric $S^1$ action having at least four isolated fixed points $p_1$, $p_2$, $p_3$, $p_4$. We will prove
Proposition 3.4.1 1. The sectional curvature of $(M^4, g)$ is zero for some two-planes.

2. $\text{Fix}(S^1, M^4) = \{p_1, \ldots, p_4\}$.

Part 1 is proved in lemma 3.4.2–3.4.4; the proof of part 2 occupies the remainder of section 3.4.

Lemma 3.4.2 For each four-tuple $1 \leq i, j, k, l \leq 4$ of distinct integers,

$$\mathcal{L}_{p_i}(p_j, p_k) + \mathcal{L}_{p_k}(p_i, p_l) + \mathcal{L}_{p_l}(p_j, p_i) \leq \pi.$$

Proof: The notation is explained in definition 2.1.4. Consider $\text{Dir}_{p_i}(p_j), \text{Dir}_{p_k}(p_i), \text{Dir}_{p_l}(p_i) \subseteq S^2 T_{p_i} M^4$. These subsets are $S^1$-invariant, so we may calculate $\mathcal{L}_{p_i}(\text{Dir}_{p_i}(p_j), \text{Dir}_{p_i}(p_k))$, etc., by passing to the metric space quotient $(S^2 T_{p_i} M^4)/S^1$. The following lemma then refutes the proof of lemma 3.4.2 to checking that if $x_1, x_2, x_3 \in S^2(\frac{1}{2}) \subseteq E^3$ then $\frac{1}{2} \left( \mathcal{L}(x_1, x_2) + \mathcal{L}(x_2, x_3) + \mathcal{L}(x_3, x_1) \right) \leq \pi$. \qed

Lemma 3.4.3 Let $S^1 \rightarrow O(4)$ be an orthogonal linear representation with no trivial direct summands, and let $(S^2(1)/S^1, \mathcal{L})$ denote the quotient space under this $S^1$ action. Then $(S^2(1)/S^1, \mathcal{L})$ admits a distance nondecreasing map to $(S^2(\frac{1}{2}), \frac{1}{2} \mathcal{L})$, where $S^2(\frac{1}{2}) = \{ x \in E^3 | |x| = \frac{1}{2} \}$, and $\frac{1}{2} \mathcal{L}$ is the distance function of the Riemannian metric on $S^2(\frac{1}{2})$ induced by the inclusion $S^2(\frac{1}{2}) \subseteq E^3$.

Proof: An orthogonal representation $S^1 \times E^4 \rightarrow E^4$ of $S^1$ is orthogonally equivalent to a representation $\Psi_{k,l} : S^1 \times C^2 \rightarrow C^2$:

$$\alpha \cdot (z_1, z_2) = (\alpha^k z_1, \alpha^l z_2)$$

where $\alpha \in S^1 \subseteq C$, $(z_1, z_2) \in C^2$, and $k, l \in Z$ are appropriately chosen. Such a representation has no fixed subspaces iff $k, l \neq 0$. Let $(\Sigma_{k,l}, \mathcal{L}_{k,l})$ denote the quotient of $S^3$ by the action $\Psi_{k,l}$.

It is easy to check that $(\Sigma_{1,1}, \mathcal{L}_{1,1})$ is isometric to $(S^2(\frac{1}{2}), \frac{1}{2} \mathcal{L})$, where $\frac{1}{2} \mathcal{L}$ is the Riemannian distance on $S^2(\frac{1}{2})$; the isometry is induced by the Hopf map $S^3(1) \rightarrow S^2(\frac{1}{2})$. 

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Let $\widehat{S^3(1)} = \{(z_1, z_2) \in C^2 \mid z_1z_2 \neq 0\}$. $\widehat{S^3(1)}$ is $\Psi_{k,l}$ invariant, and the restriction of the action $\Psi_{k,l}$ to $\widehat{S^3(1)}$ has only one orbit type, so $\widehat{S^3(1)} \cong S^3(1)/S^1$ inherits a canonical smooth submersion Riemannian metric $h_{k,l}$ from the Riemannian manifold $\widehat{S^3(1)} \subseteq E^4$. The distance function induced by this Riemannian metric coincides with $L_{k,l}|_{\Sigma_{k,l}}$. Note that $(\tilde{\Sigma}_{k,l}, h_{k,l})$ is a surface of revolution since the $T^2 = S^1 \times S^1$ action $T^2 \times C^2 \rightarrow C^2$ given by $(\alpha, \beta) \cdot (z_1, z_2) = (\alpha z_1, \beta z_2)$ induces an isometric $T^2/S^1$ action on $(\tilde{\Sigma}_{k,l}, h_{k,l})$. The meridians on $(\tilde{\Sigma}_{k,l}, h_{k,l})$ correspond to (i.e. have as inverse image) $T^2$ orbits in $\widehat{S^3(1)}$. If a given meridian has inverse image $T^2((z_1, z_2))$, $(z_1, z_2) \in \widehat{S^3(1)}$, then its length is

$$\frac{2\pi |z_1||z_2|}{\sqrt{k^2|z_1|^2 + l^2|z_2|^2}}$$

Construct a map $\tilde{\phi}_{k,l}$ from the surface of revolution $\tilde{\Sigma}_{k,l}$ to the surface of revolution $\tilde{\Sigma}_{1,1}$ which carries longitudes to longitudes, commutes with “revolution,” and preserves the correspondence of meridians with $T^2$ orbits in $\widehat{S^3(1)}$. The calculation above shows that $(\tilde{\phi}_{k,l})^* h_{3,2} \geq h_{k,l}$, so

$$\tilde{\phi}_{k,l} : (\tilde{\Sigma}_{k,l}, L_{k,l}) \rightarrow (\tilde{\Sigma}_{1,1}, L_{1,1})$$

is a distance nondecreasing map. Extending the Lipschitz map $\tilde{\phi}_{k,l}$ continuously to $\phi_{k,l} : \Sigma_{k,l} \rightarrow \Sigma_{1,1}$, we get a distance nondecreasing map

$$(\widehat{S^3(1)}/S^1, L) \approx (\Sigma_{k,l}, L_{k,l}) \rightarrow (\Sigma_{1,1}, L_{1,1}) \approx (S^2(\frac{1}{2}), L).$$

$\square$

Lemma 3.4.4

1. For each triple $1 \leq i, j, k \leq 4$ of distinct integers $L_{p_i}(p_j, p_k) + L_{p_j}(p_k, p_i) + L_{p_k}(p_i, p_j) = \pi$.

2. $(M^4, g)$ doesn't have strictly positive curvature.

3. For each quadruple of distinct integers $1 \leq i, j, k, l \leq 4$, $L_{p_i}(p_j, p_k) + L_{p_j}(p_k, p_l) + L_{p_k}(p_l, p_i) = \pi$. 

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Proof: View the four points $p_1, \ldots, p_4$ as the vertices of a tetrahedron, and add up all the angles. We have

$$\sum_{1 \leq i, j \leq 4} \sum_{j \neq i, k \neq i, k > j} \angle_{p_i}(p_j, p_k) \leq 4\pi$$

by lemma 3.4.2, while by theorem 2.1.5

$$\sum_{1 \leq i < j < k \leq 4} (\angle_{p_i}(p_j, p_k) + \angle_{p_k}(p_i, p_j) + \angle_{p_j}(p_k, p_i)) \geq 4\pi$$

Hence equality must hold everywhere, which implies the lemma. $\square$

To prove part 2 of proposition 3.4.1 indirectly, assume henceforth that $Fix(S^1, M^4)$ contains $\{p_1, \ldots, p_4\}$ strictly. From the discussion in section 3.1, $Fix(S^1, M^4)$ is a disjoint union of totally geodesic surfaces and isolated points, and section 3.3 rules out the possibility that $Fix(S^1, M^4)$ contains surfaces components. Therefore $Fix(S^1, M^4)$ must contain an additional isolated point, say $p_5$.

Lemma 3.4.5 $Fix 1 \leq i \leq 5$.

1. The sets $Dir_{p_i}(p_j) \subseteq S^2 T_{p_i} M^4$, $j \neq i$ consist of a single $S^1$ orbit each, and come in mutually orthogonal pairs.

2. The isotropy representation of $S^1$ on $T_{p_i} M^4$ is orthogonally equivalent to the action of $S^1 \subseteq C$ on $C^2$ by scalar multiplication.

Proof: For notational convenience, take $i = 5$. The reasoning of lemmas 3.4.2-3.4.4 applied to the four isolated fixed points $p_0, p_j, p_k, p_i$ implies that

$$\angle_{p_i}(p_j, p_k) + \angle_{p_i}(p_k, p_i) + \angle_{p_k}(p_i, p_j) = \pi$$

for each triple of distinct integers $1 \leq j, k, l \leq 4$. Equivalently,

$$\angle_{p_i}(Dir_{p_0}(p_j), Dir_{p_0}(p_k)) + \angle_{p_i}(Dir_{p_0}(p_k), Dir_{p_0}(p_i)) + \angle_{p_i}(Dir_{p_0}(p_i), Dir_{p_0}(p_j))$$

$$= \pi.$$
Let \( (S^3 T_{p_i} M^4)/S^1, \tilde{L}_{p_i} \) be the metric space quotient of \( S^3 T_{p_i} M^4, \tilde{L}_{p_i} \) and let \( \phi : (S^3 T_{p_i} M^4)/S^1 \to S^3(\frac{1}{2}) \) be the distance nondecreasing map provided by lemma 3.4.3. Let \( D_m = \phi \circ \pi(\text{Dir}_{p_i}(p_m)) \subseteq S^3(\frac{1}{2}), m = 1, \ldots, 4 \), where \( \pi : S^3 T_{p_i} M^4 \to S^3 T_{p_i} M^4/S^1 \) is the quotient map. For each triple \( 1 \leq i, j, k \leq 4 \) of distinct integers,

\[
\frac{1}{2} \tilde{L}(D_i, D_j) + \frac{1}{2} \tilde{L}(D_j, D_k) + \frac{1}{2} \tilde{L}(D_k, D_i) \geq \pi.
\]

This clearly forces the sets \( D_m \) to lie on a great circle, to consist of a single point each, and to pair off as antipodal points, i.e. \( \bigcup_{m=1}^4 D_m \) consists of two pairs of antipodal points. This proves part 1 of the lemma.

Part 2 of lemma 3.4.4 follows easily from lemma 2.3.1 and the fact that the isotropy representation of \( S^1 \) on \( T_{p_i} M^4 \) contains more than one pair of mutually orthogonal orbits in \( S^3 T_{p_i} M^4 \). \( \square \)

We now show that if \( \tilde{L}_{p_i}(p_j, p_k) = \frac{\pi}{2} \), then \( \text{Dir}_{p_j}(p_k) \) cannot be a single \( S^1 \)
orbit in \( S^3 T_{p_j} M^4 \), in contradiction to lemma 3.4.5. Assume—after relabelling the points if necessary—that

\[
\tilde{L}_{p_i}(p_j, p_k) = \tilde{L}_{p_i}(\text{Dir}_{p_j}(p_i), \text{Dir}_{p_j}(p_k)) = \frac{\pi}{2}.
\]

If \( \gamma_2 \in \text{Min}_{p_i}(p_j), \gamma_3 \in \text{Min}_{p_i}(p_k) \), (see definition 2.1.4) then \( \tilde{L}_{p_i}(\gamma_2(0), \gamma_3(0)) = \frac{\pi}{2} \) by lemma 3.4.5. By part 2 of theorem 2.1.5, there is a flat, totally geodesic, triangular surface \( \Delta^2 \subseteq M^4 \) with geodesic edges \( \gamma_2, \gamma_3 \), and \( \eta \in \text{Min}_{p_i}(p_j) \). If we replace \( \gamma_3 \) with \( h \cdot \gamma_3 \) for some \( h \in S^1 \), then Toponogov’s theorem will produce another flat, totally geodesic, triangular surface \( \Delta^2_h \subseteq M^4 \) with geodesic edges \( \gamma_2, h \cdot \gamma_3 \), and \( \eta_h \in \text{Min}_{p_i}(p_j) \). In particular,

\[
\tilde{L}_{p_i}(\gamma_2, \eta_h) = \tilde{L}_{p_i}(\gamma_2, \eta) = \tilde{L}_{p_i}(p_j, p_k) < \frac{\pi}{2}.
\]

Let \( v = -\gamma_2'(l_{12}) \in \text{Dir}_{p_j}(p_i) \subseteq S^3 T_{p_j} M^4 \), where \( l_{12} = \text{dist}(p_1, p_2) \), and let \( w, w_h \in \text{Dir}_{p_i}(p_k) \subseteq S^3 T_{p_i} M^4 \) be the initial velocities of \( \eta \) and \( \eta_h \), respectively. We have \( \tilde{L}_{p_i}(v, w) = \tilde{L}_{p_i}(v, w_h) = \tilde{L}_{p_i}(v, \text{Dir}_{p_i}(p_j)) < \frac{\pi}{2} \). The following lemma forces \( w = w_h \) since by lemma 3.4.5 \( \text{Dir}_{p_i}(p_k) \subseteq S^3 T_{p_i} M^4 \) consists of a single \( S^1 \)
orbit and the isotropy representation at \( p_i \) is equivalent to the standard representation \( \Psi_{1,1} \) (see lemma 3.4.3). Since \( \eta(0) = w = w_h = \eta_h(0) \) we have \( \eta_h = \eta \), which forces \( \Delta^2_h = \Delta^2 \). This is impossible unless \( h = e \in S^1 \), so we have a contradiction.
Lemma 3.4.6 Let $S^1 \subset C$ act on $C^2$ by scalar multiplication, and suppose $v, w \in S^3(1) \subset C^3$. Then either $\ell(S^1(v), S^1(w)) = \frac{\pi}{2}$ or there exists a unique $h \in S^1$ such that $\ell(v, h \cdot w) = \ell(S^1(v), S^1(w)) = \ell(v, S^1(w))$.

Proof: The Riemannian submersion metric on $S^3(1)/S^1$ is isometric to $S^2(\frac{1}{2}) \subset E^3$. The orbits $S^1(v), S^1(w) \in S^3(1)/S^1$ are either separated by $\frac{\pi}{2} = \text{diameter}(S^2(\frac{1}{2}))$, or they are joined by a unique minimizing geodesic segment $\tilde{\gamma} \subset S^3(1)/S^1$ with $\text{length}(\tilde{\gamma}) = \ell(S^1(v), S^1(w))$. If $w_0 \in S^1(w)$ satisfies $\ell(v, w_0) = \ell(S^1(v), S^1(w))$, and $\gamma$ is a minimizing geodesic segment from $v$ to $w_0$, then $\gamma$ projects to the minimizing curve $\tilde{\gamma}$. Hence $\gamma$ is the unique horizontal lift of $\tilde{\gamma}$ starting at $v$, and $w_0$ is unique. □

4 Nonnegatively Curved Riemannian Four-manifolds with $SU(2)$ Symmetry

In this section, we study the geometry of compact, one-connected, nonnegatively curved, Riemannian four-manifolds $(M^4, g)$ which admit almost effective isometric $SU(2)$ actions. We start by determining the underlying smooth $SU(2)$ manifolds.

Notation 4.0.7 $\overline{SO(2)}$, $\overline{O(2)} \subset SU(2)$ will denote $\pi^{-1}(SO(2))$ and $\pi^{-1}(O(2))$ respectively, where $\pi : SU(2) \rightarrow SO(3)$ is the usual two-fold covering homomorphism and $SO(2)$, $O(2) \subset SO(3)$ are the usual inclusions.

Smooth Classification 4.0.8 Let $(M^4, \rho)$ be a smooth, almost effective (i.e. $\ker \rho$ is discrete) $SU(2)$ action $\rho : SU(2) \times M^4 \rightarrow M^4$ on a compact one-connected four-manifold $M^4$. Then $(M^4, \rho)$ is smoothly equivalent to one of the $SU(2)$ manifolds below:

1. $(S^4, (\tilde{\rho}_3 \oplus 2\theta)|_{S^4})$. Here $\tilde{\rho}_3$ denotes the pullback of the standard three-dimensional irreducible representation $\rho_3$ of $SO(3)$ to $SU(2)$, and $\theta$ is a one-dimensional trivial representation. The orbit space $S^4/SU(2)$ is homeomorphic to $D^2$, and the orbit diagram is:
II. \((S^2 \times S^2, (\tilde{\theta} \oplus \tilde{p}_3) \big|_{S^2 \times S^2})\). \(\tilde{\theta} \oplus \tilde{p}_3\) is a representation on \(R^6\) which preserves the splitting \(R^6 \approx R^3 \oplus R^3\), so it can be restricted to \(S^2 \times S^2 \subseteq R^3 \times R^3\). \(S^2 \times S^2 / SU(2) \approx S^2\), and all the orbits are principal orbits with principal isotropy group conjugate to \(SO(2)\).

III. \((S^4, (\mu_2 \oplus \theta) \big|_{S^4})\). \(\mu_2\) is the four-dimensional irreducible representation of \(SU(2)\). \(S^4 / SU(2) \approx \{0, 1\}\). The orbit diagram is:

IV. \((CP^2, \mu_2 \oplus \theta_C)\). View \(\mu_2\) as a two-dimensional complex representation and let \(\theta_C\) be a one-dimensional trivial complex representation. Let \(\mu_2 \oplus \theta_C\) be the action induced on \(CP^2\) by the complex representation \(\mu_2 \oplus \theta_C\). The orbit diagram is:

V. \((S^4, (s^2 \tilde{p}_3 - 1) \big|_{S^4})\). \(s^2 \tilde{p}_3\) is the symmetric square of \(\tilde{p}_3\), and \(s^2 \tilde{p}_3 - 1\) is the five-dimensional irreducible summand of \(s^2 \tilde{p}_3\). The orbit diagram is:

VI. \((CP^2, (\tilde{\rho}_3)_C)\). \((\tilde{\rho}_3)_C\) is the complexification of \(\tilde{\rho}_3\), and it induces the action \((\tilde{\rho}_3)_C\) on \(CP^2\). The orbit diagram:

VII. \((SU(2) \times \phi_n S^2, \text{left})\). Let \(\phi_n : S^1 \times S^2 \to S^2\) be an \(S^1\) action with ineffective kernel \(Z_n \subseteq S^1\), \(n \geq 1\), \(Z_1 = \{e\}\). Construct the associated \(S^2\) bundle \(SU(2) \times \phi_n S^2\) to the \(S^1\) principal bundle \(S^1 \to SU(2) \to SU(2)/S^1\) and \(SU(2)/S^1 \approx CP^2\). Left translation on \(SU(2)\) induces an \(SU(2)\) action \(\text{left}\) on the associated bundle \(SU(2) \times \phi_n S^2\). The orbit diagram is:

Note that \(SU(2) \times \phi_n S^2 \approx S^2 \times S^2\) when \(n\) is even, and \(SU(2) \times \phi_n S^2 \approx CP^2\) when \(n\) is odd. To see this, observe that an \(S^2\) bundle over \(SU(2)/S^1 \approx S^2\) is determined by a transition function in \(\pi_1(SU(3)) \approx Z_2\).
Sketch of the proof of the classification: Fix an almost effective, compact, one-connected \(SU(2)\) manifold \((M^4, \rho)\). Let \(G = SU(2)\), and let \(H\) be the principal isotropy subgroup for the action \(\rho\). \(H\) is well defined up to conjugacy in \(G\). As before, \(G(p)\) and \(G_p\) denote the \(G\) orbit and the isotropy subgroup of \(p \in M^4\), respectively.

\(SU(2)\) contains no two-dimensional subgroups, so \(\text{dim}(H) = 0\) or \(\text{dim}(H) = 1\). We analyze these two cases separately.

Case A: \(\text{Dim}(H) = 1\). This gives \(H \approx SO(2)\) or \(H \approx \tilde{O}(2)\). We consider \(H \approx SO(2)\) first. Nonprincipal isotropy subgroups can only be conjugates of \(\tilde{O}(2) \subseteq SU(2)\) or \(SU(2)\) itself.

If \(G_p \approx \tilde{O}(2)\) then the slice representation of \(G_p\) on \(\nu_pG(p)\) has as principal isotropy subgroup a conjugate of \(H \approx SO(2)\). This forces the slice representation to factor through a representation of \(\tilde{O}(2)/SO(2) \approx \mathbb{Z}_2\) on \(\nu_pG(p)\). There are only two possibilities for such a \(\mathbb{Z}_2\) representation: reflection or \(-1\). The corresponding local structures for the orbit space are \(R^2/\text{reflection} \approx R \times R_+\) — a half space with \(\tilde{O}(2)\)-type orbits on the boundary, and \(R^2/\text{inversion} \approx R^2\) with an isolated \(\tilde{O}(2)\)-type orbit.

If \(G_p \approx SU(2)\) then the isotropy representation of \(SU(2)\) on \(T_pM^4\) is equivalent to \(\tilde{p}_2 \oplus \theta\) since its principal isotropy subgroup is conjugate to \(H \approx SO(2)\). Near \(G(p)\), \(M^4/SU(2)\) is homeomorphic to \(R^4/\tilde{p}_3 \oplus \theta \approx R^3/\tilde{p}_3 \times R^1 \approx R_+ \times R\), so \(M^4/G\) is locally a manifold with boundary, the orbits near \(G(p)\) being either fixed points or principal orbits.

\(M^4/G\) is a surface with boundary. Boundary components are made up of either fixed points, or \(\tilde{O}(2)\)-type orbits. We can rule out the latter possibility since the union of the \(\tilde{O}(2)\)-type orbits along a boundary component of \(M^4/G\) would be a one-sided hypersurface in \(M^4\), which is impossible since \(\pi_1(M^4) = \{e\}\).

If \(M^4\) contains two isolated \(\tilde{O}(2)\)-type orbits, then one can find a path in the surface \(M^4/G\) joining them whose inverse image in \(M^4\) is a one-sided hypersurface. So \(M^4\) contains at most one \(\tilde{O}(2)\)-type orbit. But \(M^4\) contains an even number of isolated \(\tilde{O}(2)\)-type orbits. To see this, notice that isolated \(\tilde{O}(2)\)-type orbits are in one-to-one correspondence with isolated points in \(\text{Fix} (\tilde{O}(2), M^4)\). Since \(SO(2) \triangleleft \tilde{O}(2)\), \(\tilde{O}(2)\) preserves \(\text{Fix} (SO(2), M^4)\), and
we have \( \text{Fix}(\widetilde{O}(2), M^4) = \text{Fix}(O(2), \text{Fix}(\widetilde{SO}(2), M^4)) \). Smith theory gives 
\( \chi(\text{Fix}(\widetilde{O}(2), \text{Fix}(\widetilde{SO}(2), M^4))) \equiv \chi(\text{Fix}(\widetilde{SO}(2), M^4)) \mod 2 \equiv 0 \mod 2 \) 
because \( \text{Fix}(\widetilde{SO}(2), M^4) \) is an orientable surface. But \( \text{Fix}(\widetilde{O}(2), M^4) \) is a disjoint union of circles and isolated points; therefore the number of isolated \( \widetilde{O}(2) \)-type orbits is even.

To summarize, the only nonprincipal isotropy subgroup possible is \( SU(2) \).
The only possibilities for the orbit diagram are:

because \( \pi_1(M^4/G) \) is trivial. One can easily check that these orbit diagrams determine the \( SU(2) \) manifolds up to smooth \( SU(2) \) equivalence.

Now suppose \( H \approx \widetilde{O}(2) \). The only possibility for a nonprincipal isotropy subgroup is \( SU(2) \), and this can be ruled out by observing that the isotropy representation couldn’t have the correct principal isotropy subgroup. Hence the orbits are all principal. But this forces \( M^4 \to M^4/SU(2) \) to be a trivial fiber bundle with fiber \( RP^2 \), which contradicts \( \pi_1(M^4) = \{e\} \). Hence \( H \) cannot be conjugate to \( \widetilde{O}(2) \).

Case B: \( \text{Dim}(H) = 0 \). This gives \( \text{dim}(M^4/SU(2)) = 1 \), and \( M^4/SU(2) \) is homeomorphic to \( S^1 \) or to a closed interval, depending on whether all orbits are principal or not. Since \( \pi_1(M^4/SU(2)) \) is trivial, the former case is impossible because it would give a fiber bundle \( M^4 \to S^1 \). Therefore there are two nonprincipal orbits, corresponding to the endpoints of the interval \( M^4/SU(2) \). Neither of these can be an exceptional orbit, because that would produce a one-sided hypersurface in \( M^4 \). So the nonprincipal orbits \( G(p_1), G(p_2) \) are both singular, with \( G_{p_1} \) conjugate to \( \widetilde{SO}(2), O(2), \) or \( SU(2) \).

If \( G_{p_1} \) is conjugate to \( \widetilde{SO}(2) \), then \( \text{dim}(\nu_{p_1} G(p_1)) = 2 \) and the slice representation of \( G_{p_1} \) is fixed by its principal isotropy subgroup \( \approx Z_n \).

If \( G_{p_1} \approx O(2), \) then \( \text{dim}(\nu_{p_1} G(p_1)) = 2 \), and one can check that the slice representation of \( G_{p_1} \) on \( \nu_{p_1} G(p_1) \) must factor through the two-fold covering homomorphism \( \pi \big|_{\widetilde{O}(2)} : \widetilde{O}(2) \to O(2) \). The two-dimensional representations of \( O(2) \) with discrete kernel are indexed by their dihedral principal isotropy groups \( D_n \subseteq O(2) \). We denote the corresponding principal isotropy subgroups in \( O(2) \) by \( \widetilde{D}_n \).

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If \( G_{p_1} = SU(2) \) then the isotropy representation must be \( \approx \mu_2 \) since \( \dim(H) = 0 \). Hence \( H = \{e\} \).

Now we consider the possibilities for the triple \((H, G_{p_1}, G_{p_2})\). If \( G_{p_1} = SU(2) \) then \( H = \{e\} \), which forces \( G_{p_2} = SU(2) \) or \( G_{p_2} = SO(2) \) with \( G_{p_2} \) acting effectively on the slice \( \nu_{p_1}G(p_1) \). Both of these possibilities occur as actions III and IV.

If \( G_{p_1} \) is conjugate to \( \widetilde{O}(2) \), then \( H \) is conjugate to \( \tilde{D}_n \), so either \( G_{p_2} \) is conjugate to \( \widetilde{O}(2) \) or \( G_{p_2} \) is conjugate to \( SO(2) \) and \( n = 1 \). These possibilities are realized in actions V and VI. When \( G_{p_2} \) is conjugate to \( \widetilde{O}(2) \), the only way to achieve \( \pi_1(M^4) = \{e\} \) is to have \( H \) conjugate to \( \tilde{D}_2 \).

All of the actions above are determined up to smooth \( SU(2) \) equivalence by the triple \((H, G_{p_1}, G_{p_2})\), together with the condition that \( \pi_1(M^4) = \{e\} \).

\[ \square \]

All of the \( SU(2) \) manifolds in the classification above admit obvious positively curved invariant metrics except those of type II and VII. The results of section 3 imply that \( SU(2) \) manifolds of type II and VII do not admit such metrics, but we give a direct proof here. We will show that an invariant metric on an \( SU(2) \) manifold of type II or VII carries disjoint totally geodesic surfaces. Theorem 2.1.1 then implies that such metrics don't have positive curvature.

Consider first an invariant metric on an \( SU(2) \) manifold of type II. \( Fix(S\widetilde{O}(2), S^2 \times S^2) \) is a disjoint union of two surfaces, and it must be totally geodesic in \( S^2 \times S^2 \) relative to any \( SU(2) \)-invariant metric.

Now consider an invariant metric on an \( SU(2) \) manifold of type VII. The two singular orbits are totally geodesic provided \( n \neq 4 \). If \( G(p) \) is one of the singular orbits, then \( G_p \) is conjugate to \( SO(2) \). Denote the representation of \( G_p \) on \( R^2 \) with ineffective kernel \( Z_n \) by \( R^2_{(n)} \). The isotropy representation of \( G_p \) on \( T_pG(p) \) is equivalent to \( R^2_{(2)} \), while the slice representation of \( G_p \) on \( \nu_{p}G(p) \) is equivalent to \( R^2_{(1)} \). The second fundamental form of \( G(p) \) at \( p, II_p \) is invariant under the action of \( G_p \), i.e.

\[
II_p \in Hom_{G_p}(s^2(T_pG(p)), \nu_pG(p))
\]

\[ \approx Hom_{G_p}(s^2(R^2_{(2)}), R^2_{(n)}) \]

\[ \approx Hom_{G_p}(R^1 \oplus R^2_{(1)}, R^2_{(n)}) \]
\[ \approx \text{Hom}_{\mathcal{G}_n}(R^1, R^2_{(n)}) \oplus \text{Hom}_{\mathcal{G}_n}(R^2_{(n)}, R^1_{(n)}) \approx \{0\} \]

unless \( n = 4 \).

On the other hand, if \( n > 2 \) then \( \text{Fix}(H, M^4) \) is a surface with two components. Again, this is totally geodesic with respect to any invariant metric.

**Theorem 4.0.9** Let \( g \) be an \( SU(2) \)-invariant metric on an \( SU(2) \)-manifold of type II. If \( g \) has nonnegative curvature, then the \( SU(2) \)-Riemannian manifold \( (S^2 \times S^2, (\theta \oplus \varphi))_{|S^2 \times S^2}, g) \) is \( SU(2) \)-equivalent to a Riemannian product of \( (S^2, g_1) \) and \( (S^2, g_2) \), where \( g_1 \) has nonnegative curvature, \( g_2 \) has constant curvature, and \( SU(2) \) acts on the second factor of the Riemannian product \( (S^2, g_1) \times (S^2, g_2) \).

**Proof:** We will view this action as an effective \( SO(3) \) action on \( S^2 \times S^2 \). Let \( g \) be an \( SO(3) \)-invariant metric on \( S^2 \times S^2 \) with the action \( h \cdot (x, y) = (x, h \cdot y) \), \( (x, y) \in S^2 \times S^2 \), \( h \in SO(3) \). To show that \( g \) is a product metric, we show that the horizontal subspaces \( \pi_2^{-1}(y) \), \( y \in S^2 \) intersect the vertical subspaces \( \pi_1^{-1}(x) \), \( x \in S^2 \), orthogonally, and we find metrics \( g_1 \) and \( g_2 \) on \( S^2 \) so that \( \pi_i : (S^2 \times S^2, g) \rightarrow (S^2, g_i) \), \( i = 1, 2 \) are Riemannian submersions.

If \( p = (x, y) \in S^2 \times S^2 \), then \( (SO(3))_p \approx SO(2) \) preserves the submanifolds \( S^2 \times \{ y \} = \pi_2^{-1}(y) \) and \( \pi_1^{-1}(x) \), so the isotropy representation of \( (SO(3))_p \) preserves \( T_p \pi_2^{-1}(y) \) and \( T_p \pi_1^{-1}(x) \). Having no \( (SO(3))_p \) irreducible summands in common, these two subspaces are \( g(p) \) orthogonal, i.e. \( \pi_2^{-1}(y) \) intersects \( \pi_1^{-1}(x) \) orthogonally.

Identify \( \pi_1 : S^2 \times S^2 \rightarrow S^2 \) with the quotient map \( S^2 \times S^2 \rightarrow (S^2 \times S^2)/SO(3) \approx S^2 \times (S^2/\text{SO}(3)) \approx S^2 \) and let \( g_1 \) be the submersion metric on \( S^2 \) induced by \( \pi_1 \). Then clearly \( \pi_1 : (S^2 \times S^2, g) \rightarrow (S^2, g_1) \) is a Riemannian submersion.

It remains only to show that there is a metric \( g_2 \) on \( S^2 \) such that \( \pi_2 : (S^2 \times S^2, g) \rightarrow (S^2, g_2) \) is a Riemannian submersion. To prove this it is sufficient to show that the fibers \( \pi_2^{-1}(y_1) \) and \( \pi_2^{-1}(y_2) \) are equidistant for nearby \( y_1 \) and \( y_2 \), i.e. \( \text{dist}(\cdot, \pi_2^{-1}(y_2)) \) and \( \text{dist}(\pi_2^{-1}(y_1), \cdot) \) are constant functions on \( \pi_2^{-1}(y_1) \) and \( \pi_2^{-1}(y_2) \) respectively.
Note: $g$ is invariant under the $O(3)$ action $h \cdot (x, y) = (x, h \cdot y)$, $(x, y) \in S^2 \times S^2$, $h \in O(3)$.

Proof of Note: The $O(3)$ orbits are the same as the $SO(3)$ orbits, so it suffices to check that $(O(3))_p \approx O(2)$ preserves $g(p) \in s^2(T^*_p(S^2 \times S^2))$ for every $p \in S^2 \times S^2$. But this follows easily from the fact that $(SO(3))_p \approx SO(2)$ preserves $g(p)$.

Given $y_1, y_2 \in S^2$ there is a reflection $r \in O(3)$ which fixes $y_1$ and $y_2$. Therefore $Fix(r, S^2 \times S^2) = S^2 \times Fix(r, S^2) \cong S^2 \times S^2$ is a totally geodesic, nonnegatively curved submanifold containing $\pi_2^{-1}(y_1)$ and $\pi_2^{-1}(y_2)$. Lemma 2.1.3 shows that the totally geodesic surfaces $\pi_2^{-1}(y_1)$ and $\pi_2^{-1}(y_2)$ must be equidistant inside $Fix(r, S^2 \times S^2)$. It follows that $\pi_2^{-1}(y_2)$ and $\pi_2^{-1}(y_2)$ are equidistant in $S^2 \times S^2$ if $y_1$ and $y_2$ are close enough. $\Box$

We now investigate the structure of nonnegatively curved invariant metrics on $SU(2)$ manifolds of type VII. Recall that $n$ is the order of the principal isotropy subgroup of this action.

Let $G(q_1)$ and $G(q_2)$ be the two singular orbits and let $l = dist(G(q_1), G(q_2))$. Assume the $q_i \in G(q_i)$ have been chosen so that $dist(q_1, q_2) = l$.

**Lemma 4.0.10** Let $\gamma : [0, l] \longrightarrow M^4$ be an $SU(2)$-geodesic segment with $\gamma(0) = q_1$, $\gamma(l) = q_2$. If $n > 2$ then parallel transport along $Im\gamma$ maps $T_{q_1}G(q_1)$ to $T_{q_2}G(q_2)$.

**Proof**: Let $G_{\gamma} = \{ h \in G | h \text{ fixes } Im(\gamma) \}$. The uniqueness of parallel transport along $Im\gamma$ implies that parallel transport along $\gamma$ gives an orthogonal transformation $P_{\alpha} : T_{\gamma(0)}M^4 \longrightarrow T_{\gamma(0)}M^4$ which is equivariant with respect to the isotropy representations of $G_\gamma$ on $T_{q_1}M^4$ and $T_{q_2}M^4$. For all $n$, $G_{\gamma} = \{ h \in G | h \gamma(0) = \gamma(0) \}$ is a conjugate of $H$, while for $n > 2$ the isotropy representations of $G_\gamma$ on $T_{q_1}M^4$ and $T_{q_2}M^4$ have fixed subspaces $\nu_{q_1}G(q_1)$ and $\nu_{q_2}G(q_2)$ respectively. This forces $P_{\alpha}(T_{q_1}G(q_1)) = T_{q_2}G(q_2)$.

Alternatively, note that if $n > 2$, then $N^2 = Fix(G_\gamma, M^4)$ is a disjoint union of two totally geodesic $S^2$'s, one of which contains $Im\gamma$. Hence parallel transport along $Im\gamma$ preserves $TN^2$ and $\nu N^2$. Since $N^2$ intersects $G(q_1)$ and $G(q_2)$ orthogonally, $P_{\alpha}(T_{q_1}G(q_1)) = T_{q_2}G(q_2)$. $\Box$

**Lemma 4.0.11** Let $q_1, q_2$, and $\gamma$ be as in lemma 4.0.10. Then

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1. Any parallel field $X : [0, 1] \rightarrow \nu(\text{Im}\gamma)$ with $X(0) \in T_0 G(q_1)$ is the restriction to $\text{Im}\gamma$ of an $SU(2)$- Killing vector field.

2. Any such $X$ is orthogonal to $\exp(D_t \nu_0, G(q_1)) = G_\theta(\text{Im}\gamma) = \cup_{h \in G_h} h \cdot \text{Im}\gamma$.

Proof: Consider the family of equidistant hypersurfaces $N_t = \{x \in M^4 | \text{dist}(x, G(q_1)) = t\}, t \in (0, 1)$. $\text{Im}\gamma$ is normal to this family, so we can apply the Riccati equation to the second fundamental forms of the $N_t$ along $\text{Im}\gamma$. This implies, for $t \in (0, 1), X : [0, 1] \rightarrow \nu(\text{Im}\gamma), X$ parallel along $\text{Im}\gamma$, that 

$$\frac{d}{dt}(g(II_{N_t}(X(t), X(t)), \gamma'(t))) \geq 0$$

with equality only if $II_{N_t}(X(t), Y) = 0$ for all $Y \in \nu_{x(t)} \text{Im}\gamma$.

If $X(0) \in T_0 G(q_1)$, then $X(l) \in T_0 G(q_2)$, so

$$g(II_{G(q_1)}(X(0), X(0), \gamma'(0))$$

$$= \lim_{t \to 0} g(II_{N_t}(X(t), X(t)), \gamma'(t))$$

$$\leq \lim_{t \to 0} g(II_{N_t}(X(t), X(t)), \gamma'(t))$$

$$= g(II_{G(q_2)}(X(l), X(l)), \gamma'(l)).$$

If we take $X_1(0), X_2(0) \in T_0 G(q_1)$ orthonormal, then $X_1(l), X_2(l) \in T_0 G(q_2)$ will be orthonormal and $0 = g(H_{G(q_1)}, \gamma'(0))$

$$= g(\sum_{i=1,2} II_{G(q_1)}(X_i(0), X_i(0)), \gamma'(0))$$

$$\leq g(\sum_{i=1,2} II_{G(q_2)}(X_i(l), X_i(l)), \gamma'(l))$$

$$= g(H_{G(q_2)}, \gamma'(l)) = 0$$

where $H_{G(q_1)}$ denotes the mean curvature of the orbit $G(q_1)$, and the singular orbits $G(q_1)$ and $G(q_2)$ are minimal. This forces

$$\diamond \quad II_{N_t}(X_i(t), Y) = 0$$

for all $t \in (0, 1), i = 1, 2, Y \in \nu_{x(t)} \text{Im}\gamma$.

A vector field $X : (0, 1) \rightarrow \nu(\text{Im}\gamma)$ along $\text{Im}\gamma$ is a Jacobi field corresponding to a variation through geodesics normal to the $N_t$ if and only if

$$\nabla_{\gamma'(t)} X = W(t) X(t)$$

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where $W(t) : T_{r(t)}N_t \to T_{r(t)}N_t$ is the Weingarten map of $N_t$, or equivalently, if

$$g(\nabla_{\gamma(t)}X, Y) = -g(I\gamma_t(X(t), Y(t), \gamma'(t))$$

for every $t \in (0, l)$, $Y \in \nu_{r(t)}Im\gamma$. From $\circ$ it is clear that the $X_i$, $i = 1, 2$, are Jacobi fields along $Im\gamma$ corresponding to variations through geodesics normal to the $X_t$, and since $SU(2)$ acts transitively on $N_t$ for each $t$, the $X_i$ are induced by Killing fields.

We now show that the $X_i$ are normal to the disk $\exp(D_{i\nu_{r(t)}G(q_t)}) = G_{q_t}(\gamma([0, l]))$. Pick $t_0 \in (0, l)$, and solve for $Y : (0, l) \to \nu(Im\gamma)$ with $g(Y(t_0), X_i(t)) = 0$, $|Y(t_0)| \neq 0$, and $(\nabla_{\gamma(t)}Y)(t) = W(t)Y(t)$. Then as before, $Y$ is the restriction to $Im\gamma$ of an $SU(2)$-Killing field on $M^4$. What's more, $g(Y(t), X_i(t)) = 0$ for every $t \in (0, l)$ since $\frac{d}{dt}(g(Y(t), X(t)))$

$$= g(\nabla_{\gamma(t)}X, X_i(t)) = -g(I\gamma_t(Y(t), X_i(t)), \gamma'(t)) = 0.$$

Hence the Killing field which restricts to $Y$ is zero at $q_1$—for otherwise the Killing fields which restrict to $Y$, $X_1$, $X_2$ would span a three-dimensional subspace of $T_{q_1}M^4$—and must therefore arise from the action of the isotropy group $G_{q_t}$. Away from $q_1$, $T(G_{q_t}(Im\gamma))|_{Im\gamma}$ is spanned by $T(Im\gamma)$ and $Y$, so the $X_i$ are in fact normal to $G_{q_t}(Im\gamma)$.

We can now prove

**Theorem 4.0.12** If $n > 2$, and $g$ is a nonnegatively curved invariant metric on a $SU(2)$-manifold $M^4$ of type VII with principal isotropy subgroup $Z_n$, then $(M^4, g)$ admits an $SU(2)$-equivariant Riemannian submersion to $S^3$ with totally geodesic fibers.

**Proof:** Keeping the notation from the last few lemmas, $\exp|D_{t\nu G(q_t)} : D_{t\nu G(q_t)} \to M^4 \setminus G(q_2)$ is a diffeomorphism since the $SU(2)$-geodesic segments given by $t \mapsto \exp(tv)$, $v \in \nu G(q_1)$, $|v| = 1$, are $SU(2)$-minimizing on the interval $[0, l]$. The bundle projection $D_{t}(\nu G(q_1)) \to G(q_1)$ therefore induces a $C^\infty$ submersion $\tilde{\pi}_1 : M^4 \setminus G(q_2) \to G(q_1)$. $\tilde{\pi}_1$ is $SU(2)$-equivariant, so in order to show that it is a Riemannian submersion, we need only check the Riemannian submersion property along $\gamma([0, l]) \subset M^4 \setminus G(q_2)$, where $\gamma$ is the same as in the preceding lemmas. Pick $t_0 \in (0, l)$ and $X(t_0) \in \nu_{r(t)}\tilde{\pi}_1^{-1}(q_1) = \ldots$

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By the previous lemma, $X(t_0)$ extends to a Killing field which is parallel along $\text{Im}\gamma$ and normal to $G_{\gamma}(\gamma([0,t])) = \bar{\pi}_1^{-1}(q_1)$. As before, call the restriction of this Killing field to $\text{Im}\gamma, X$. We have $|\bar{\pi}_1 X(t_0)| = |X(0)| = |X(t_0)|$ since $\bar{\pi}_1$ is $SU(2)$-equivariant. Hence $\bar{\pi}_1$ is orthogonal on the fiber normals, and $\bar{\pi}_1$ is a Riemannian submersion.

In fact, $\bar{\pi}_1 : M^4 \setminus G(q_2) \longrightarrow G(q_1)$ extends to a $C^\infty$ submersion $\pi : M^4 \longrightarrow G(q_1)$. To see this, observe that if $\bar{\pi}_2 : M^4 \setminus G(q_1) \longrightarrow G(q_2)$ is the Riemannian submersion obtained by imitating the construction of $\bar{\pi}_1$ but with $G(q_1)$ replaced by $G(q_2)$, then the fibers of $\bar{\pi}_2$ coincide with the fibers of $\bar{\pi}_1$ in $(M^4 \setminus G(q_2)) \cap (M^4 \setminus G(q_1)) = M^4 \setminus (G(q_1) \cup G(q_2))$. We can smoothly identify $G(q_1)$ and $G(q_2)$ as the quotient space of $M^4 \setminus (G(q_1) \cup G(q_2))$ by the same smooth relation, i.e. there is a diffeomorphism $\phi : G(q_2) \longrightarrow G(q_1)$ so that

$$\phi \circ \bar{\pi}_2\big|_{M^4 \setminus (G(q_1) \cup G(q_2))} : M^4 \setminus (G(q_1) \cup G(q_2)) \longrightarrow G(q_1)$$

$$= \bar{\pi}_1\big|_{M^4 \setminus (G(q_1) \cup G(q_2))}$$

The submersions $\bar{\pi}_1$ and $\phi \circ \bar{\pi}_2$ can then be glued along $M^4 \setminus (G(q_1) \cup G(q_2))$ to give a smooth Riemannian submersion $\pi : M^4 \longrightarrow G(q_1)$. □

References


