Research Statement

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This research statement provides an overview of my research activity in two different areas of harmonic analysis: oscillatory integrals and singular integral operators.

In the line of oscillatory integrals, I am interested in obtaining sharp and stable decay rate estimates of oscillatory integrals and related problems in various settings, including, but not limited to: the scalar setting, the operator setting and, in particular, the multilinear setting. My research in the direction of singular integral operators, lying in the intersection of singular Radon-like transforms and multilinear operators, concerns the boundedness of the bilinear Hilbert transform along curves and its maximal function analogues.

1 Oscillatory integrals

Oscillatory integrals have long been of interest in harmonic analysis and mathematical physics. Lately, they have emerged as powerful analytic tools in various problems, ranging from PDEs to geometry and number theory.

Scalar oscillatory integrals, commonly referred to as oscillatory integrals of the first kind, concern the asymptotic behavior of

\[ I(\lambda, S) = \int_{\mathbb{R}^n} e^{i\lambda S(x)} \phi(x) dx \quad \text{as} \quad |\lambda| \to \infty, \]  

(1)

where \( \phi \) is a smooth cut-off and the phase function \( S \) is real-valued. The operator analogues are often referred to as the second kind. For instance, the (1+1)-dimensional case can be formulated as

\[ T_\lambda(f)(x) = \int_{\mathbb{R}} e^{i\lambda S(x,y)} \phi(x, y) f(y) dy. \]  

(2)

The theme is to understand the decay rate estimates of the norms of these operators. The theory for one-dimensional scalar oscillatory integrals is well-established thanks to the van der Corput lemma (see [32]). However, in dimensions larger than one the theory is a lot less complete and the progress has been slow, because, among many other reasons, the singularities and the geometry of the phases involved may themselves be substantially more complicated. Indeed, in dimensions higher than two, Varchenko [34] identified a polynomial phase whose optimal decay rate for the corresponding scalar oscillatory integral is unstable under small perturbations, meaning that a small deformation of the phase may lead to worse decay rates. Consequently, there can’t be a unified theory for scalar oscillatory integrals in dimensions higher than two. However, in dimensions two, Varchenko [34] is able to obtain sharp decay rate estimates for arbitrary real analytic phases and a result of Karpushkin [17] guarantees their local stability. In the direction of oscillatory integral operators, Phong and Stein [26, 27, 28, 29] developed a systematic method that can efficiently handle the (1+1)-dimensional case, as well as some higher-dimensional case (joint with Sturm [30]). In particular, Phong and Stein [28] proved the sharp \( L^2 \)-estimates for \( T_\lambda \).
Another problem that has a strong impact on my research is the multilinear oscillatory integral forms introduced by Chrest, Li, Tao and Thiele [5]:

\[ M(S, \Pi) = \int_{\mathbb{R}^N} e^{i \lambda S(x)} \phi(x) \prod_{j=1}^J f_j(\pi_j(x)) \, dx. \]  

(3)

Here the pair of inputs \([S, \Pi]\) consists of a real-valued function \(S\) and \(\Pi = \{\pi_j\}_{1 \leq j \leq J}\), where each \(\pi_j: \mathbb{R}^N \to \mathbb{R}^n\) is a surjective linear transformation. Under what conditions on the input \([\Pi, S]\) do there exist \(\delta > 0\) and \(C < \infty\) such that \(|M(S, \Pi)| \leq C|\lambda|^{-\delta} \prod_j \|f_j\|_\infty\) for all \(\lambda \in \mathbb{R}\)? A more refined question is to find the optimal exponent \(\delta\) about the estimates with \(\prod_j \|f_j\|_\infty\) replaced by \(\prod_j \|f_j\|_{p_j}\). To date little is known about the general multilinear case for both questions, though significant progress has been made in [5, 3, 4, 7].

**Trilinear oscillatory integral forms and stability.** A significant part of my research is the study of the following trilinear form

\[ \Lambda_S(f_1, f_2, f_3) = \int_{\mathbb{R}^2} e^{i \lambda S(x,y)} \phi(x,y) f_1(x) f_2(y) f_3(x+y) \, dx dy. \]  

(4)

Being a particular case of [3], \(\Lambda_S\) can also be considered as a trilinear extension of the dual of \(T_\lambda\). However, \(\Lambda_S\) is different from \(T_\lambda\) due to an extra convolution structure (both operators contain an oscillation structure). As observed in [5], one expects no decay at all when \(S(x,y)\) may be annihilated by \(D = \partial_x \partial_y (\partial_x - \partial_y)\). Let \(j_0\) denote the degree of a lowest degree term in the Taylor expansion of \(DS\) at the origin and define the relative multiplicity of \(S\) to be the number \(n_S = (j_0 + 3)\).

**Theorem 1** ([35]) Let \(S\) be real analytic in the support of \(\phi\). Then

\[ |\Lambda_S(f_1, f_2, f_3)| \leq C|\lambda|^{-\frac{1}{2n_S}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2. \]  

(5)

The proof of this theorem extends the framework of [28] to the trilinear setting, relying on two new ingredients: a trilinear extension of Phong-Stein’s operator van der Corput lemma and an algorithm for two-dimensional resolution of singularities. The algorithm, rooted in the Newton-Puiseux algorithm (see [8] for example) and influenced by [34], [28] and [14], may be of independent interest. It can be used to decompose a neighborhood of an isolated singular point into finitely many subregions, on which derivatives of the phase behave like monomials. Many problems dealing with real analytic functions can be reduced to the ones dealing essentially with monomials.

Scaling arguments show that the exponent of \(\lambda\) cannot be improved in (1), but a comparison to the sublevel set estimate indicates that the decay rate \(|\lambda|^{-1/2n_S}\) is likely not the best possible if one considers \(L^p\) spaces on the right-hand side other than \(L^2\). One reason is that the oscillation structure is not quite exploited in (5). In joint work with Gressman [15], we successfully utilized both structures, improving the decay rate from \(|\lambda|^{-1/2n_S}\) to \(|\lambda|^{-\frac{2}{n_S}}\) for generic phases.

**Theorem 2** ([15]) Let \(S(x,y)\) be as above and \(n_S \geq 9\). If the order of “vanishing” of \(DS\) is less than \((\frac{n_S}{2} - 2)\), then

\[ |\Lambda_S(f_1, f_2, f_3)| \leq C|\lambda|^{-\frac{2}{n_S}} \|f_1\|_\infty \|f_2\|_\infty \|f_3\|_\infty. \]  

(6)
While this matching of decay rates is satisfying and natural, it should perhaps be regarded as somewhat surprising that this is possible since, among other things, the highest possible decay rate can only be achieved when $f_1, f_2$ and $f_3$ all belong to $L^\infty$, which is not traditionally a regime in which one expects to find strong cancellation effects. This theorem implies some interesting stable estimates. Under the same assumption, there is a constant $C$ such that

$$\left| \int \int_{\mathbb{R}^2} e^{i(\lambda S(x,y) + P_1(x) + P_2(y) + P_3(x+y))} \phi(x, y) dx dy \right| \leq C |\lambda|^{-\frac{2}{7}}$$

(7)

uniformly for all real-valued measurable functions $P_1, P_2$ and $P_3$. Analogue result for the sublevel set estimate also holds. In particular, stable estimates for the oscillatory integrals under quadratic perturbations and sharp estimates for the Fourier transform of the surface measure of the two-dimensional manifold $(x, y, x^2, y^2, xy, S(x, y))$ follow by appropriate choices of $P_1, P_2$ and $P_3$ in [7].

Oscillatory integral operators and van der Corput lemma. Another important part of my ongoing research activity is dedicated to the study of the oscillatory integral operators $T_\lambda$. In [36], I show that for all real analytic phases the $L^p \to L^p$ mapping properties of the corresponding $T_\lambda$ can be fully characterized by the Newton polyhedron. Let $S(x, y) = \sum_{k,l \in \mathbb{N}} c_{k,l} x^k y^l$ be the Taylor expansion at the origin. The reduced Newton polyhedron $N^*(S)$ of $S$ is the convex hull of the union of all $[k, \infty) \times [l, \infty)$ with $c_{k,l} \neq 0$ and $k, l \geq 1$. Let $D^*(S)$ be the boundary of $N^*(S)$, and $\delta$, the Newton distance, be the number such that $(\delta, \delta) \in D^*(S)$.

**Theorem 3 ([36])** Let $S$ and $\phi$ be as above. Assume $\alpha > 0$ and $\phi(0, 0) \neq 0$. Then

$$\|T_\lambda\|_{p \to p} \leq C |\lambda|^{-\alpha} \iff \left( \frac{1}{\alpha p}, \frac{1}{\alpha p} \right) \in N^*(S)$$

and this estimate is sharp if and only if $(\frac{1}{\alpha p}, \frac{1}{\alpha p}) \in D^*(S)$.

This theorem implies, in particular, the fundamental work of Phong and Stein [28], which corresponds to the $p = 2$ and $\alpha = \delta$ case. An equivalent form of this theorem is connected to a two-dimensional analogue of the van der Corput lemma:

**Theorem 4 ([36])** Let $k$ and $l$ be two positive integers and $K \subset \mathbb{R}^2$ be any compact set. Assume $\phi \in C_0^\infty(K)$ and $S$ is real analytic on $K$ with $\partial^k_x \partial^l_y S(x, y) \neq 0$ for all $(x, y) \in K$. Then there is a constant $C$ such that

$$\|T_\lambda\|_{k^l \to \frac{k+l}{k}} \leq C |\lambda|^{-\frac{1}{k+l}}, \quad \text{for all } \lambda \in \mathbb{R}. \quad (8)$$

Multilinear oscillatory integrals and stability. In [34], Varchenko proved sharp estimates of $[1]$ for analytic phases satisfying the so-called Varchenko’s non-degenerate condition. One natural operator analogue of $[1]$ in higher dimensions is the following multilinear form:

$$\Lambda(f_1, \ldots, f_d) = \int_{\mathbb{R}^d} e^{i\lambda S(x)} \phi(x) \prod_{j=1}^d f_j(x_j) dx,$$

(9)

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $S(x) = \sum_{\alpha} c_\alpha x^\alpha$ is a real analytic function. Let $N^*(S)$ denote the reduced Newton polyhedron of $S$ and $\delta$ denote the corresponding Newton distance. Under the
assumption $S$ is polynomial-like, Phong, Stein and Sturm \cite{30} and Carbery and Wright \cite{1} obtain nearly sharp estimates for $\Lambda$: for each $\alpha \in \mathbb{N}^*$ and for $1/p_j = 1 - \alpha_j/|\alpha|$, $|\Lambda(f_1, \ldots, f_d)| \lesssim \log |\lambda|^{-1/|\alpha|} \prod |f_j|_{p_j}$. It turns out that the behavior of (9) is substantially richer than anticipated by these earlier results. In a joint program with Gilula and Gressman, we prove the following interesting results:

**Theorem 5** \cite{11} Suppose $S$ is real analytic and satisfies a second order of Varchenko’s non-degenerate condition. If $p_j \in [2, \infty]$, for $1 \leq j \leq d$, then

$$|\Lambda(f_1, \ldots, f_d)| \lesssim \log |\lambda|^{-\frac{1}{|\alpha|}} \prod_{j=1}^d |f_j|_{p_j}.$$  

is true for some $\nu > 2$ if and only if $(\frac{\nu}{p_1'}, \ldots, \frac{\nu}{p_d'}) \in \mathbb{N}^*(S)$.

Let the number $k$ denote the dimension of the main face of $\mathbb{N}^*(S)$. By taking $p_j = \infty$ and $f_j(x_j) = e^{iP_j(x_j)}$, the theorem implies the following sharp and stable estimates:

$$\left| \int \frac{e^{i\lambda S(x) - \sum_{j=1}^d P_j(x_j)}}{\lambda} \phi(x) dx \right| \leq C \log(2 + |\lambda|)^{d-1-k} |\lambda|^{-\frac{1}{2}},$$

where the constant $C$ is uniform over all measurable functions $P_j$ and the exponent of the log term is also optimal.

Finally, the author (jointly with Gressman) is currently working on the oscillatory Loomis-Whitney inequalities. Let $\{\pi_j\}_{1 \leq j \leq d}$ be the Loomis-Whitney input, namely, each $\pi_j : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ is the projection that deletes the $j$-th component. Consider estimates of the oscillatory Loomis-Whitney form

$$\left| \int_{\mathbb{R}^d} e^{i\lambda S(x)} \phi(x) \prod_{j=1}^d f_j(\pi_j(x)) dx \right| \lesssim |\lambda|^{-\alpha} \prod_{j=1}^d |f_j|_{p_j}.$$  

(11)

Surprisingly, in dimensions larger than two, sharp estimates of this form are completely unknown in any case. Considering estimates of the form (11) can be employed to deduce a rich class of stability results, any progress towards this problem will be of great significance.

**Future plan.** There are many interesting questions need to be answered in this line. What are the full range estimates of $\Lambda_S$, $T_\lambda$ and $\Lambda$? Can the above results be extended to the smooth setting (namely, the phase function is assumed to be smooth rather than analytic) or the weighted setting or the higher-dimensional setting? Can one strengthen (8) to uniform estimates, obtaining a full version of the two-dimensional van der Corput lemma? Theorem 5 also suggests a completely new approach to obtain stability results – by proving multilinear $L^\infty$-estimates. One exciting problem to work on is the sharp $L^\infty$-estimates of the oscillatory Loomis-Whitney inequalities (11).

In sum, my long-term goal in this line is to advance the understanding of oscillatory integrals in various settings of higher dimensions. This requires, in each fixed setting, developing suitable analytic tools to tackle the non-singular case and a suitable algorithm for resolution of singularities (in a very broad sense) to address the more singular case.
2 Singular integral operators

Another main theme of my research lies in the intersection of singular Radon-like transforms and multilinear singular operators. Radon-like transforms and singular variants arise naturally in many different areas of analysis and geometry. The theory of singular Radon-like transforms developed over a period of nearly half century, beginning from Fabes [9], advancing by the efforts of many mathematicians (see [31] for a survey) and reaching a zenith in the work of Christ, Nagel, Stein, and Wainger [6], who demonstrated that singular Radon-like transforms are bounded on $L^p$ for $1 < p < \infty$ subject to a weak geometric curvature (finite-type) condition. The theory of multilinear singular integrals with modulation symmetries, of which the bilinear Hilbert transform (BHT) is a basic example, is relatively young and has a history of about two decades. The basic framework of this theory, commonly referred to as time-frequency analysis, dates back to the celebrated proof of the pointwise convergence of Fourier series first by Carleson [2] and later by Fefferman [10], and was more systematically developed by Lacey and Thiele [18, 19] in their proof of the boundedness of the BHT.

Motivated by these two theories, Li [21] initialized the study of the bilinear Hilbert transform along curve (BHTaC), namely, the following operator,

$$H_\gamma(f,g)(x) = \text{p.v.} \int_{\mathbb{R}} f(x-t)g(x - \gamma(t)) \frac{dt}{t},$$

(12)

where $\gamma : \mathbb{R} \to \mathbb{R}$ is a smooth function. There is no doubt that curvature plays a fundamental role. In the linear setting, at least in the translation invariant case, a straightforward way to exploit curvature and thus to build up the $L^2$-theory, is to couple Plancherel’s theorem with various devices from oscillatory integrals (the method of stationary phase, van der Corput lemma and etc.); see [6] for a distinct approach. In the bilinear setting, however, the role of curvature is a lot more involved and quite difficult to exploit due to the lack of $L^2$-theory. In [21], Li managed to do by embedding the bilinear operator into a trilinear oscillatory form and using a new technique called the $\sigma$-uniformity, inspired by [12], establishing the $L^2 \times L^2 \to L^1$ boundedness of $H_\gamma$ when $\gamma$ is a monomial. In joint work with Li [23], we settled the full $L^p \times L^q \to L^r$ ranges for $H_\gamma$ and its maximal analogue $M_\gamma$ (see [20]) except for the endpoint case when $\gamma$ is any “non-flat” polynomial. These full-range estimates, characterized in terms of contact orders, are an indication of a more subtle role of geometry in the theory of bilinear (and multilinear) Radon-like transforms.

Theorem 6 ([23]) If $\gamma(t)$ is a polynomial without linear term, then the following are equivalent:

- $H_\gamma$ and $M_\gamma$ map from $L^p \times L^q$ to $L^r$ for all $r > \frac{k-1}{k}$, $p, q > 1$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$;
- All the roots of $\gamma'(t) - 1 = 0$ have order at most $(k - 1)$;
- For any tangent line $L$ of $(t, \gamma(t))$ with slope equal to 1, the contact order between $L$ and $(t, \gamma(t))$ is at most $k$.
- There is a constant $C$ s.t. the following sublevel set estimate is true for $h$ sufficiently small:

$$|\{t : |\gamma'(t) - 1| < h\}| < Ch\frac{1}{k-1}.$$
The polynomial assumption on $\gamma$ is not essential. When $\gamma(t) = |t|^\alpha$ ($\alpha > 0$ and $\neq 1$), for instance, $H_\gamma$ and $M_\gamma$ map into $L^r$ for $r > 1/2$, providing some hope for the conjecture that the classical BHT also maps into $L^r$ in the same range. Our approach also extends the uniform estimates for the BHT \cite{33, 13, 22} to the curvature setting:

**Theorem 7** \cite{23} If $\gamma(t)$ is a polynomial of degree $d$ without linear term, then for all $r > \frac{d-1}{d}$, $H_\gamma$ and $M_\gamma$ map into $L^r$ uniformly, in the sense the bounds are independent of the coefficients of $\gamma$.

In joint work with Guo \cite{16}, we further investigate the role of curvature in the boundedness of the BHTaC. In the linear setting, Nagel, Vance, Wainger, and Weinberg \cite{25} proved that such boundedness is captured by the auxiliary function $h(t) = t\gamma'(t) - \gamma(t)$ (assuming $\gamma(0) = 0$). For example, when $\gamma(t)$ is odd and convex, the corresponding Hilbert transform is bounded if and only if $h(t)$ has a bounded double time property \cite{25}. Our investigation suggest that the bilinear analogue of this auxiliary function should be the quotient between $\gamma(t)$ and $t\gamma'(t)$. More precisely, for each dyadic number $\epsilon > 0$ and $|t| \in [1/4, 4]$, let $q_\epsilon(t) = (\epsilon^\gamma(t))^{-1}\gamma(\epsilon t)$. Then the key curvature condition to obtain a bound for $H_\gamma$ is that both $|q''_\epsilon(t)|$ and $|(q''_\epsilon(t))^2 - q'\epsilon(t)q''\epsilon(t)|$ can be uniformly bounded from below. A similar result was previously obtained by Lie \cite{24}.

**Future plan.** Inspired by \cite{25}, it is natural to ask what are the necessary and sufficient conditions on $\gamma$ (assumed to be convex) such that $H_\gamma$ is bounded from $L^2 \times L^2 \to L^1$? The first step towards this problem is to prove such boundedness when $\gamma$ is asymptotically flat, namely, $\gamma$ is a polynomial containing both linear and non-linear terms. The new challenge lies in the interaction between the linear and non-linear terms in the phase plane, making the operator behave quite differently from both the BHT and the BHTaC. Note that such phenomenon is new in the bilinear setting and this line of investigation deserves further effort.

To date the theory of multilinear singular Radon-like transforms is still in its early stage and very little is known. The classical $L^2$-theory, though works perfectly well in the linear setting, becomes increasingly inefficient as the level of the multi-linearity of the transform increases. Besides, the lack of suitable multilinear analogue of Cotlar-Stein makes the exploitation of orthogonality a lot harder in the multilinear setting. Completely novel ideas are necessary to understand the role of curvature and to develop suitable analytic tools for the theory of multilinear operators. There are promising signs that advances in the theory of multilinear oscillatory integrals (and multi-parameter oscillatory integrals) can eventually push forward the development of multilinear singular Radon-like transforms.

**References Cited**


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