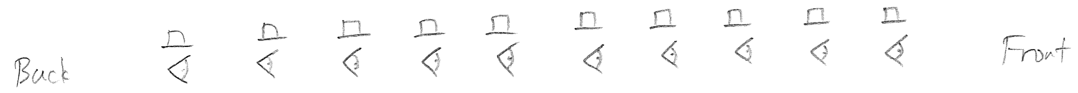
Math Academy Recitation 2017-07-17

Warmup: Finite Hat Problem

There are 10 prisoners and 10 hats, each colored red or blue. A warden lines up the prisoners in a single line and (randomly) assigns a hat to a prioner without the prisoner knowing the color of anyone's hat. Now the prisoners are forced to face one end of the line, so they can only see the color of the hats in front of them.



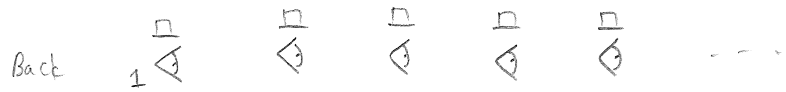
starting from the back, each prisoner has to guess the color of one's own hat out loud, and is killed immediately if they guess it wrong. The warden gives them a chance to discuss a strategy before the "game" starts. What is a most optimal strategy?

↳ We can guarantee all but the backmost prisoners to not be killed, with the survival rate of the backmost prisoner being 50%. This is the strategy: The prisoners all agree that if the backmost prisoner sees an odd number of red, he says "red", else he says "blue".

One can also think of the following, probably only useful in math and theoretical physics.

Infinte Hat Problem

the same setup, except we have countably infinite many prisoners with a "well-defined" backmost prisoner, a.k.a. prisoner 1, i.e.



the goal now is to find a strategy that guarantees survival of all but finitely many prisoners, given the prisoner can say exactly one word (maybe not red/blue), and:

(a) The guess of each prisoner is made known to everyone.

(b) The guess of each prisoner is only made known to the warden.

⌜ Aside | Countability:

We said "countably infinite" because it allows us to arrange the prisoners. In general set S is said to be <u>countable</u> if there is an injection $S \overset{i}{\hookrightarrow} \mathbb{Z}_{\geq 1}$. Clearly any subset T of a countable set S is still countable: use $T \hookrightarrow S \overset{i}{\hookrightarrow} \mathbb{Z}_{\geq 1}$.

<u>Example:</u> $\mathbb{Z}$ is countable by the following function.

$$f: \mathbb{Z} \to \mathbb{Z}_{\geq 1} \quad \text{by} \quad x \mapsto \begin{cases} 1 & \text{if } x = 0 \\ 2x & \text{if } x \geq 1 \\ 2|x|+1 & \text{if } x \leq -1 \end{cases}$$

<u>Example:</u> $\mathbb{Q}$ is countable as it can be viewed as a subset of $\mathbb{Z}^2$, and this is countable as $\mathbb{Z}^2 \overset{f \times f}{\hookrightarrow} \mathbb{Z}_{\geq 1}^2$.

<u>Example:</u> $\mathbb{R}$ is <u>uncountable</u>. The proof is the famous diagonalization argument. It suffices to consider the subset $D := \{ 0.a_1a_2a_3 \dots : a_i \text{ is either } 0 \text{ or } 1 \}$. Suppose that D is countable, so we can arrange elements of D. Consider the number $0.x_1x_2x_3\dots$ in D, where

$$x_i = \begin{cases} 0 & \text{if the } i^{\text{th}} \text{ decimal place in the } i^{\text{th}} \text{ number of } D \text{ is } 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then $0.x_1x_2x_3\dots$ is not among the elements of D, a contradiction.

<u>Example:</u> The set $\mathbb{I}$ of irrational numbers in $\mathbb{R}$ is uncountable. If it were countable, then we have $\mathbb{R} = \mathbb{Q} \amalg \mathbb{I}$ is countable, a contradiction.

This is made more complicated by the fact that the visual representations of $\mathbb{R}, \mathbb{Q}, \mathbb{I}$ are all the same: ———————. More about this later. ⌟

Solving the infinite hat problem uses the Axiom of Choice:

<u>Axiom of Choice</u> (AoC): Let $X = \{X_i\}_{i \in I}$ a multiset of nonempty sets. We can always find a choice function for X, i.e. a function $f: X \to \bigcup_{i \in I} X_i$ such that $f(X_i) \in X_i$ for each $i \in I$.

Informally, the Axiom of Choice is the following: If we have a collection of nonempty baskets filled with balls, we can pick a ball from each basket. Sounds logical enough right? We will discuss why it is, and why it isn't. But first, let us solve the infinite hat problem.

Solution to (a). We can actually guarantee all but the backmost prisoners to not be killed, with the survival rate of the backmost prisoner being 50%.

This is the strategy. Let S be the set of all possible hat arrangments on all but prisoner 1. Partition S into sets X_i, where two elements belong to the same X_i if the hat colors only differ finitely many times. Let $X = \{X_i\}_{i \in I}$. By AoC, there is a choice function $f: X \to \bigcup_{i \in I} X_i$. Call $f(X_i)$ the representative of X_i. Notice the correct hat sequence is in one of the sets X_i, say it is X_1.

Now we declare each element in X_1 to be either 0 or 1 as follow. Fix $f(X_1)$ to be 0, and let an element $x \in X_1$ to be

$$\begin{cases} 0 & \text{if the number of hats differ from } f(X_1) \text{ an even number of times} \\ 1 & \text{otherwise.} \end{cases}$$

We are now ready. The backmost prisoner uses his information of the hats to realize that the arrangement must be in X_1, then say the number attached to it. The second prisoner also knows the hat arrangement is one of two in X_1, labeled 0 and 1. Based on whether prisoner 1 survived s/he will know the hat color. Proceed similarly for the other prisoners. □

Solution to (b). We cannot guarantee how many will survive, just finitely many may die. Use the same choice function f as above, and have each prisoner say the corresponding color of $f(X_1)$. □

Exercise on AoC: Here is another game, with persons A and B.

1. A thinks of any function $F: \mathbb{R} \to \mathbb{R}$ (with "Lebesgue measure").
2. B randomly picks any point $p \in \mathbb{R}$.
3. A tells B the values $F(x)$ for any $x \neq p$.
4. B guesses the value of $F(p)$ strategically, given this information.

The assertion is that B can guess $F(p)$ with probability one. WHY??

Solution. B partitions the set $S = \{\text{functions } f: \mathbb{R} \to \mathbb{R}\}$ into X_i, such that two functions are in the same X_i if they differ at only finitely many points.

By AoC there is a choice function $c: \{X_i\}_{i \in I} \to \bigcup_{i \in I} X_i$. Say that F lies in X_1.

After step 3, B determines that F must lie in X_1 and simply read out the value of $c(X_1)$ at p. As p was chosen randomly on the real line, the probability that it lands on one of the finitely many points where $c(X_1)$ differs from F is 0. $\square$

↳ Note: In step 2 we could have requested B to pick finitely many points.

[Aside] Really what was important above is that a finite set of points on $\mathbb{R}$ has measure zero. A measure is basically a formal way to assign volume to a subset of a space, provided it is possible. ⟨Insert informal definition of Lebesgue measure here⟩. For example, $\mathbb{Q}$ has measure 0! The Cantor set has positive measure though it has empty interior, so measure is not quite the "volume" we imagine. There are also nonmeasurable sets, or sets we cannot meaningfully assign "volume" to. Here is an example. Partition the unit interval $[0,1]$ into sets X_i, with p and q belonging to the same X_i if $p - q \in \mathbb{Q}$. By AoC there is a choice function

$c : \{X_i\}_{i \in I} \to \bigcup_{i \in I} X_i$. Our nonmeasurable set will be $S = \{ c(X_i) : i \in I \}$.

It is nonmeasurable as $\coprod_{q \in \mathbb{Q} \cap [-1,1]} (S + q) \subset [-1, 2]$, and all $S + q$ has the same measure if it were measurable, but $[-1,2]$ has finite measure 3. ┘

We now argue some reasons for or against AoC.

<u>For AoC</u>:

It should be widely believed that the following are true:

<u>Law of excluded middle (LEM)</u>: For every proposition P, either P or not P (not both).

<u>Obvious fact</u>: Subsets of finite sets are finite.

┌ <u>Aside</u> | In fact, let us list two easy applications of LEM.

<u>Russell's Paradox</u>: There is no sets of all sets.

<u>Proof</u>. Suppose U were the set of all sets. Consider $R = \{ x \in U : x \notin x \}$. We get a contradiction: if $R \in R$, then $R \notin R$; if $R \notin R$, then $R \in R$. □

<u>Fact</u>: There are irrational numbers a, b with a^b rational.

<u>Indirect Proof</u>. Either $(\sqrt{2})^{\sqrt{2}}$ is rational, or $((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} = 2$. □

<u>Direct Proof</u>. $(\sqrt{2})^{\log_2 9} = \sqrt{2^{\log_2 9}} = \sqrt{9} = 3$. □ ┘

We will show how AoC is related to these statements.

<u>Thm</u>: (a) AoC implies LEM

(b) LEM is equivalent to the fact that subsets of finite sets are finite

<u>Proof</u>. (a) Let P be a proposition. We need to decide P. Let

$A := \{ x \in \{0,1\} : P \vee (x = 0) \}$ and $B := \{ x \in \{0,1\} : P \vee (x = 1) \}$.

Notice $0 \in A$ and $1 \in B$. By AoC there is a choice function $c: \{A, B\} \to A \cup B$.

Notice $c(A) \in A$, $c(B) \in B$, and $A \cup B \subset \{0,1\}$.

(1) If $c(A) = 1$, then $1 \in A$, so necessary $P \vee (1=0)$, and P.

(2) If $c(B) = 0$, then $0 \in A$, so necessarily $P \vee (0=1)$, and P.

(3) If $c(A) = 0$ and $c(B) = 1$, then not P, else if P then we will have $A = B = \{0,1\}$, and so $0 = c(A) = c(B) = 1$, a contradiction. (Of course we have $0 \neq 1$).

Notice (3) cannot happen if we have (1) or (2), though the choice function is not unique.

(b) Assume LEM. Let $A = \{a_1, \dots a_n\}$ be a finite set, and $B \subseteq A$ a subset. For each a_i, we can decide if $a_i \in B$ or $a_i \notin B$, so we can also enumerate B, hence finite.

Assume that subsets of finite sets are finite. Let P be a proposition. We need to decide P. Let $A = \{0\}$ and $B = \{x \in A : P\}$. Since $B \subseteq A$, it is finite as well, so we can enumerate $B = \{b_0, \dots b_m\}$. If $m=0$ then P, else not P. $\square$

Against AoC: A discussion on dissection of polytopes

Consider any two polyhedra P_1 and P_2. We can ask the following problem.

(I) Can we cut P_1 into finitely many polyhedral pieces and reassemble to form P_2? [Hilbert's Third Problem]

(II) What if the pieces can be just any set?

Clearly (I) works with special cases of P_1 and P_2. Let us give the answers.

Theorem I. A cube and a tetrahedron of the same volume does not satisfy (I).
(Max Dehn)

Theorem II. If P_1 and P_2 are any bounded sets with nonempty interior in $\mathbb{R}^3$, then we can dissect P_1 into finitely many pieces to form P_2.
(Banach-Tarski)

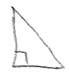
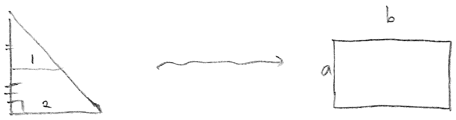
↳ For example, theoretically we can cut up a ball and form two balls, both identical to the original ball, and we can cut up a pea to form another Sun. The pieces are not measurable though.

Theorem II relies on the <u>Axiom of Choice</u>. This is a theorem that explains why AoC creates a lot of controversy among logicians.

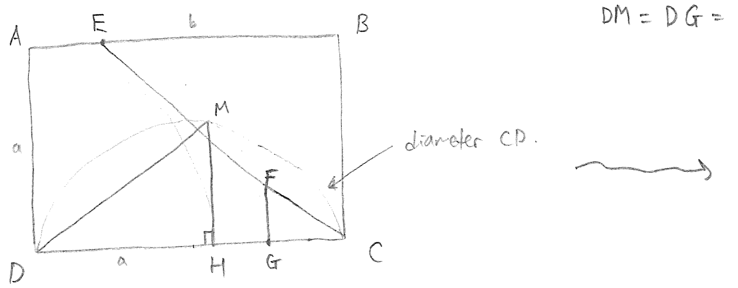
We now explain how to prove Theorems I and II. But before that let us mention the following

<u>Easy Result</u>. Let P_1 and P_2 be two polygons with same area. Then we can cut P_1 into finitely many polygonal pieces to form P_2.

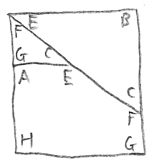
<u>Proof</u>. Fix an area, and let S be the set of all polygons with this area. We can partition S into sets X_i, where two polygons are in the same X_i if they satisfy the conditions of the easy result (Explain why.). Hence it suffices to assume P_2 is a square. Now do the following.

- Triangulate P_1 to get a bunch of triangles.
- Cut each triangle into two to form right triangles. 
- 

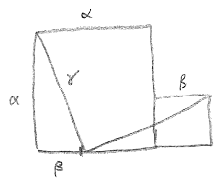
rectangle. (done if square) or if $b = ka$, $k \in \mathbb{Z}_{\geq 1}$
- By cutting up, can assume $a < b < 2a$.
- $DM = DG = BE$



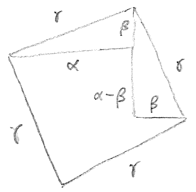
square!


So we get a bunch of squares.

- To reassemble a bunch of squares to form a big squares, use Pythagoras' Theorem:



$\alpha^2 + \beta^2 = \gamma^2$



bigger square!

$\square$

Proof of Theorem I.

The idea is to construct an additive invariant on polyhedra. This will be an assignment to any polyhedra P a number $D(P) \in \mathbb{R}$ such that

$$D(P_1 \sqcup P_2 \sqcup \cdots \sqcup P_n) = D(P_1) + D(P_2) + \ldots + D(P_n).$$

If such an assignment D is defined, then any P_1, P_2 satisfying (I) must have $D(P_1) = D(P_2)$. In particular we just need to find such a D with $D(\text{cube}) \neq D(\text{tetrahedron})$.

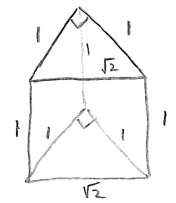
uses Zorn's Lemma ($\Leftrightarrow$ Axiom of Choice)

Definition of Dehn's invariant. Consider any $\mathbb{Q}$-basis of $\mathbb{R}$ containing π, and pick a $\mathbb{Q}$-linear map $f: \mathbb{R} \to \mathbb{R}$ with $f(\pi) = 0$ (might need to explain this).

Dehn's invariant (with respect to f) is defined as follow: For a polyhedron P,

$$\boxed{D(P) := \sum_{e \in \{\text{edges of } P\}} \left(\text{length of } e\right) \cdot f\left(\text{dihedral angle of } e\right)}$$

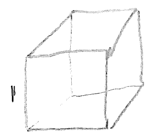
Example.

$P =$ 

$$\begin{aligned} D(P) &= 5 f\left(\tfrac{\pi}{2}\right) + 2\sqrt{2}\, f\left(\tfrac{\pi}{2}\right) + 2 f\left(\tfrac{\pi}{4}\right) \\ &= \left(\tfrac{5}{2} + \sqrt{2} + \tfrac{1}{2}\right) f(\pi) \\ &= 0. \end{aligned}$$

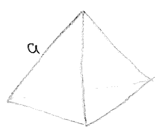
Easy do-at-home exercise. Dehn's invariant is additive. □

It remains to compute $D(\text{cube})$ and $D(\text{tetrahedron})$, with volumes 1, say.



$$D(\text{cube}) = 12 \cdot f\left(\tfrac{\pi}{2}\right) = 6 f(\pi) = 0$$

$$D(\text{tetrahedron}) = 6a f\left(\arccos\left(\tfrac{1}{3}\right)\right).$$



volume $= \dfrac{a^3}{6\sqrt{2}}$

dihedral angle $= \arccos\left(\tfrac{1}{3}\right)$

Question: Can we choose f so that $f(\arccos(\tfrac{1}{3})) \neq 0$? In other words, we need $\arccos(\tfrac{1}{3})$ $\mathbb{Q}$-linearly independent to π, or equivalently $\cos(\tfrac{p}{q}\pi) \neq \tfrac{1}{3}$ for any $p, q \in \mathbb{Z}$.

Claim: $\cos\left(\frac{p}{q}\pi\right) \neq \frac{1}{3}$ for any $p, q \in \mathbb{Z}$.

Proof. Suppose $\cos\left(\frac{p}{q}\pi\right) = \frac{1}{3}$. Then Euler's formula gives

$$e^{i\frac{p}{q}\pi} = \cos\left(\tfrac{p}{q}\pi\right) + i\sin\left(\tfrac{p}{q}\pi\right) = \frac{1}{3} \pm i\,\frac{2\sqrt{2}}{3}.$$

(Last equality uses $\sin^2\theta + \cos^2\theta = 1$). Assume $e^{i\frac{p}{q}\pi} = \frac{1}{3} + i\,\frac{2\sqrt{2}}{3}$; the other case is similar.

We have $\quad 1 = e^{i2p\pi} = \left(\frac{1}{3} + i\,\frac{2\sqrt{2}}{3}\right)^{2p}$.

↳ Use induction to see that $\left(\frac{1}{3} + i\,\frac{2\sqrt{2}}{3}\right)^n = \frac{a_n}{3^{n+1}} + i\sqrt{2}\,\frac{b_n}{3^{n+1}}$, with

$$a_n = a_{n-1} - 4b_{n-1}, \qquad b_n = b_{n-1} + 2a_{n-1}. \qquad (a_n, b_n \text{ integers})$$

Hence we just need to show $b_{2p} \neq 0$. It suffices to show $b_{2p} \not\equiv 0 \pmod 3$. We have $a_1 = 1$ and $b_1 = 2$, so $a_2 = -7 \equiv 2 \pmod 3$ and $b_2 = 4 \equiv 1 \pmod 3$, so $a_3 \equiv 1 \pmod 3$ and $b_3 \equiv 2 \pmod 3$, etc ... $\square$

Proof of Theorem II

(Need to explain notations as necessary.)

To simplify notation let us give the following

Definition. Let G be a group acting on a set X. Say that $A, B \subseteq X$ are <u>G-equidecomposable</u> if $A = \coprod_{\text{finite}} A_i$, $B = \coprod_{\text{finite}} B_i$, and such that $B_i = g_i A_i$ for some $g_i \in G$.

In our application to prove Banach-Tarski paradox, G will be a special group of rotations in $\mathbb{R}^3$, and $X = \mathbb{R}^3$. Before doing this though, we will need to go off-tangent a little and talk about the ...

<u>Banach-Cantor-Schröder-Bernstein (BCSB) theorem</u>. This is an important theorem creating partial orders from well-behaved equivalence relations.

BCSB theorem. Let $\sim$ be an equivalence relations on sets satisfying:

(1). If $A \sim B$ then there is a bijection $b : A \xrightarrow{\cong} B$ such that $A' \sim b(A')$ whenever $A' \subseteq A$.

(2). If $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$ with $A_1 \sim B_1$ and $A_2 \cup B_2$, then $A_1 \cup A_2 \sim B_1 \cup B_2$.

Then the partial order $\prec$, defined by $A \prec B$ if $A \sim B'$ for some $B' \subset B$, is a partial order.

Proof. Clearly $\prec$ is reflexive and transitive. It suffices to prove antisymmetry. Suppose $A \prec B$ and $B \prec A$. Then there exists $f : A \xrightarrow{\cong} B_1$, $g : A_1 \to B$, for $B_1 \subset B$ and $A_1 \subset A$, and f, g satisfying (1). Define C_n as follow:

$$C_0 = A \setminus A_1 \quad , \quad \text{and} \quad C_{n+1} = g^{-1} f(C_n).$$

Let $C = \bigcup_{n=0}^{\infty} C_n$. Then $g(A \setminus C) = B \setminus f(C)$, so $A \setminus C \sim B \setminus f(C)$. But also $f \sim f(C)$, and hence using (2) we get $(A \setminus C) \cup C \sim (B \setminus f(C)) \cup f(C)$, thus $A \sim B$. $\square$

Examples of BCSB: • Works when $\sim$ is equidecomposability

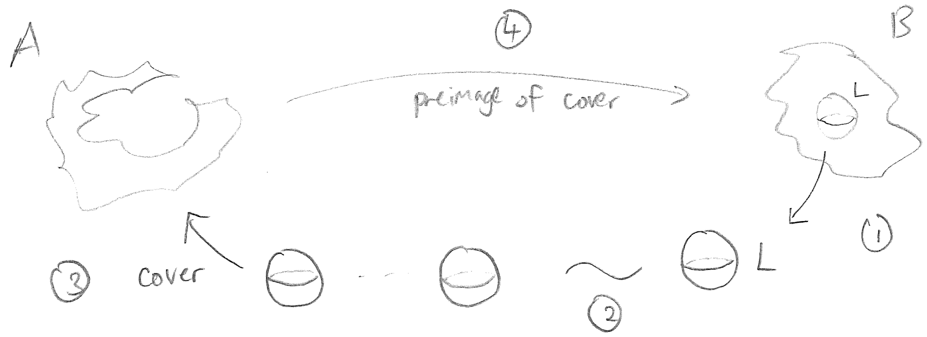
• Works when $\sim$ is cardinality. (This is the usual Cantor-Bernstein theorem)

In the rest of these notes we concentrate on proving the

Weak Banach-Tarski Paradox. The unit ball is equidecomposable with two copies of itself.

But let us first see how this implies the Banach-Tarski paradox.

Proof of Banach-Tarski (Theorem II): By BCSB theorem it suffices to prove P_1 is equidecomposable with a subset of P_2. The visual proof is as follow.



In words, let L be a ball inside B, and use Weak Banach-Tarski to reproduce finitely many of them by equidecomposability to cover A. Then take the preimage of A inside the many copies of L to get a subset of B that A is equidecomposable with (this last step requires a slighty more delicate argument using finiteness. □

We now prove the Weak Banach-Tarski paradox to complete the proof of Theorem II.

<u>Lemma 1</u>. Let F be the free group on 2 generators, and let $G = F \times \mathbb{Z}/2\mathbb{Z}$, where we write $\mathbb{Z}/2\mathbb{Z} = \{1, r\}$ with $r^2 = 1$ Then $F \times \{1\}$ is G-equidecomposable with all of G (where G acts on itself by left translation).

↳ As an easy consequence, F is F-equidecomposable with two subsets of itself that partitions it there is a partition $F = F_1 \sqcup F_2$, with each F_i F-equidecomposable with F.

<u>Proof</u>. Again use BCSB. $F \times \{1\} \prec G$ via the identity map. For the other direction, write $F = \mathrm{Free}\{\sigma, \tau\}$, and define

$$W(\rho) := \{\text{words that begin with } \rho \text{ in shortest form}\}.$$

Then

$$G = \bigcup \begin{cases} G_1 = W(\sigma) \times \{1\} \\ G_2 = (F \setminus W(\sigma)) \times \{1\} \\ G_3 = W(\tau) \times \{r\} \\ G_4 = (F \setminus W(\tau)) \times \{r\} \end{cases} \rightsquigarrow \left.\begin{array}{l} F_1 = (1,1) G_1 = W(\sigma) \times \{1\} \\ F_2 = (\sigma^{-1}, 1) G_2 = W(\sigma^{-1}) \times \{1\} \\ F_3 = (1, r) G_3 = W(\tau) \times \{1\} \\ F_4 = (\tau^{-1}, r) G_4 = W(\tau^{-1}) \times \{1\} \end{array}\right\} \text{disjoint subsets of } F \times \{1\}$$

. □

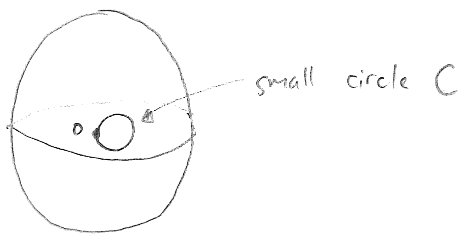
Lemma 2: There is a free group F on two generators generated by two rotations in $\mathbb{R}^3$.

Proof Sketch. Rotation by x-axis, and by z-axis, is such an option. One can show any nontrivial word in F sends $(0,1,0)$ to $\frac{(a, b\sqrt{2}, c)}{3^k}$, $b \not\equiv 0 \pmod 3$. □

Now always let $\mathbb{B}$ be the unit ball, and F the free group of Lemma 2.

Lemma 3. $\mathbb{B}$ is equidecomposable with $\mathbb{B} \setminus \{0\}$.

Proof. Consider a small circle passing through O and contained entirely in $\mathbb{B}$.



Let ρ be a 1-radian rotation of C. Then $S := \{0, \rho \cdot 0, \rho^2 \cdot 0, \dots\}$ are distinct points. Applying ρ to all these points gets $\{\rho \cdot 0, \rho^2 \cdot 0, \dots\} = S \setminus \{0\}$. Thus the equidecomposition is

$\mathbb{B} = S \sqcup (\mathbb{B} \setminus S)$, with $\rho \cdot S = S \setminus \{0\}$ and $1 \cdot (\mathbb{B} \setminus S) = \mathbb{B} \setminus S$. □

Lemma 4. Let D be the set of points on the sphere S^2 (note: not $\mathbb{B}$!) fixed by some nontrivial element of F. Then S^2 is equidecomposable with $S^2 \setminus D$.

Proof. Note that D is countable because F is a countable set, and each nontrivial element of F fixes two points. Thus we can find an axis a of S^2 that does not intersect D. Note that there exist a rotation $\rho \notin F$ such that $\rho(D) \cap D = \emptyset$ (as $D \times D$ is countable). Hence $\rho^n(D) \cap D = \emptyset$ as well for $n > 0$, else if $d = \rho^n(e)$ for some $d, e \in D$, then $\rho^{n-1}(e) \in D$ (as $\rho^{n-1} \notin F$) so that $d = \rho(\rho^{n-1}(e)) \in \rho(D)$. Thus the orbits $S = \{D, \rho D, \rho^2 \cdot D, \dots\}$ are disjoint. As in Lemma 4, the equidecomposition is $S^2 = D \sqcup (S^2 \setminus D)$. □

<u>Lemma 5</u>: $S^2 \setminus D$ can be partioned into two sets, each of which is equidecomposable with $S^2 \setminus D$.

<u>Proof</u>. Note that any two $f_1, f_2 \in F$ sends any point in X to two different images by definition of D. Hence the sets $F \cdot x$, $x \in S^2 \setminus D$, partitions $S^2 \setminus D$. By the AoC, there is a choice function $c : \{Fx\}_{x \in S^2 \setminus D} \longrightarrow \coprod_{x \in S^2 \setminus D} Fx$, i.e. a set M consisting of exactly one member from each Fx. In other words, each $x \in S^2 \setminus D$ can be uniquely written as $x = fm$ for some $f \in F$ and $m \in M$.

By Lemma 1 we have $F = F_1 \amalg F_2$ with each F_i F-equidecomposable with F. Thus $F_1 M = \{ f_1 m : f_1 \in F, m \in M \}$ and $F_2 M$ is a partition of $FM = S^2 \setminus D$.
Note that there is an equidecomposition between $F_i M$ and FM; if $A_{ik} \subseteq F$ is a piece of F_i that is mapped bijectively to $B_{ik} \subset F$ via some $g_{ik} \in F$, then g_{ik} sends $A_{ik} \cdot M$ to $B_{ik} \cdot M$. $\square$

<u>Proof of the Weak Banach-Tarski paradox</u>

Notice $S^2 \underset{\text{Lemma 4}}{\sim} S^2 \setminus D \overset{\text{Lemma 5}}{\sim} (S^2 \setminus D) \amalg (S^2 \setminus D) \overset{\text{Lemma 4}}{\sim} S^2 \amalg S^2$.

Viewing $\mathbb{B} \setminus \{0\} = \coprod_{0 < r < 1} (\text{sphere of radius } r)$, we see that

~~to~~ $\mathbb{B} \setminus \{0\}$ is equidecomposable with $(\mathbb{B} \setminus \{0\}) \amalg (\mathbb{B} \setminus \{0\})$.

Applying Lemma 3 to both sides gives the result !!! $\square$

<u>A remark</u>. The Banach-Tarski paradox tells us that the sets in the construction must <u>not</u> be Lebesgue measurable.

⌈Side Comment. The Banach-Tarski Paradox does not work in two dimensions. ⌋

While we are on the subject of balls and spheres, here is a classical puzzle --

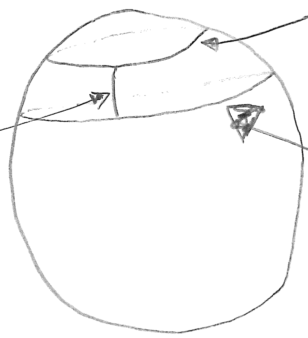
Assume the Earth is a perfect sphere. How many points are there on Earth where you can:

- walk one mile,
- turn right 90° and walk one mile,
- turn right 90° and walk another mile,

and end up in the same spot?

Options: 1, 2, finitely many, countably infinitely many, uncountably infinitely many

Answer: Uncountably infinitely many! Here is an uncountable number of starting points:



References :

(1) Denis Bashkirov, slides on "Hilbert's Third Problem and Dehn's Invariant".

(2) Andrej Bauer, "Five stages of Accepting Constructive Mathematics".

(3) Anders Kaseorg, "The Banach-Tarski Paradox".

(4) Stan Wagon, "The Banach-Tarski Paradox"