Multizeta values in characteristic p

"I study special values of ζ"

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Classical case

$$\zeta(s_1,\ldots,s_n)=\sum_{a_1>\cdots>a_n\geq 1}\frac{1}{a_1^{s_1}\cdots a_n^{s_n}}\qquad (s_1\geq 2, s_i\geq 1)$$

- Known analytic continuation! (Known functional equation of $\zeta(s)$!)
- Interpretation as periods of "Mixed Tate Motives"
- Actual values of multizeta? Hard.

$$\zeta(2n)=\frac{(-1)^{n+1}2^{2n-1}B_{2n}\pi^{2n}}{(2n)!}$$

$$\zeta(2,\ldots,2)=\zeta(2^{[n]})=\frac{\pi^{2n}}{(2n+1)!}$$

- Algebraic relations? Many known! Here are two simple ones.

$$\zeta(2,1)=\zeta(3)$$
$$\zeta(a)\zeta(b)=\zeta(a,b)+\zeta(b,a)+\zeta(a+b)$$

- Algebraic independence? Nothing known!

Is $\zeta(3)$ transcendental?

Characteristic p case

$$\zeta(s_1,\dots,s_n) = \sum_{\substack{a_i \in \mathbb{F}_q[\theta] \text{ monic} \\ \deg a_1 > \cdots > \deg a_n \geq 0}} \frac{1}{a_1^{s_1} \cdots a_n^{s_n}} \qquad (s_i \geq 1)$$

- Unknown analytic continuation! (Unknown functional equation of $\zeta(s)$!)
- Interpretation as periods of "Mixed Carlitz Motives"
- Actual values of multizeta? Hard.

$$\zeta((q-1)s) = \frac{B_{(q-1)s}}{\Gamma_{(q-1)s+1}} \tilde{\pi}^{(q-1)s}$$

- Algebraic relations? Not much known. Here is the shuffle relation.

$$\begin{aligned}\zeta(a)\zeta(b) &= \zeta(a,b) + \zeta(b,a) + \zeta(a+b) \\ &\quad + \sum_{\substack{i+j=a+b \\ q-1 \mid j}} \left((-1)^{a-1} \binom{j-1}{a-1} + (-1)^{b-1} \binom{j-1}{b-1} \right) \zeta(i,j)\end{aligned}$$

- Algebraic independence? Known for $\zeta(s)$ and some families of multizeta!

Algebraic independence of $\zeta(s)$ in characteristic p

Theorem (Chang-Yu 2007)

Each of the values $\zeta(1), \zeta(2), \ldots$ is transcendental over k. Furthermore, the set

$$\{\tilde{\pi}, \zeta(s) : q-1 \nmid s \text{ and } p \nmid s\}$$

is algebraically independent over $\overline{k}$, and all algebraic relations between the values $\zeta(1), \zeta(2), \ldots$ are generated by the relations

$$\zeta((q-1)s) = \frac{B_{(q-1)s}}{\Gamma_{(q-1)s+1}} \tilde{\pi}^{(q-1)s}, \qquad \zeta(ps) = \zeta(s)^p.$$

Key ideas

- Write $\zeta(s)$ as $\mathbb{F}_q[\theta]$-linear sums of "polylogarithms".
- Realize the "polylogarithms" as periods of Mixed Carlitz Motives.
- Study motivic Galois groups of these t-motives (exists by Tannakian).

Similar ideas can be used for multizeta values, though we can't achieve a complete description of algebraic independence (yet).

My main work: Generalize these to *colored multizeta values* (i.e. multizeta values attached with certain Hecke characters).