Cartier divisors

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1 Examples

Example 1.1. Let X be the affine quadric cone $X = \operatorname{Spec} k[X,Y,Z]/(XY - Z^2)$. We will show $CaCl = 0$ and $Cl = \mathbb{Z}/2$.

Example 1.2. Let X be a double cover of $\mathbb{P}^3$ ramified in the Barth sextic surface:

$$4(\tau^2x^2 - y^2)(\tau^2y^2 - z^2)(\tau^2z^2 - x^2) - \omega^2(1+2\tau)(x^2+y^2+z^2-w^2)^2 = 0$$

in $\mathbb{P}^4$. Here $\tau = \frac{1+\sqrt{5}}{2}$. This is a rational threefold. $CaCl(X) = \mathbb{Z}$, $Cl(X) = \mathbb{Z}^{14}$. See example 1.4 in [CJ].

Remark 1.3. For a normal variety, if an integer multiple of a Weil divisor is a Cartier divisor, we call it a $\mathbb{Q}$-Cartier divisor. We can define operations on Cartier divisors and extend by linearality to $\mathbb{Q}$-Cartier divisors. (up to numerical equivalence?). If on a normal scheme every Weil divisor is $\mathbb{Q}$-Cartier, we call it Q-factorial. It is easy to see Example 1.2 is not Q-factorial. See Definition 2.12 [McK]. $\diamond$

Remark 1.4. It is often easy to do computations in toric varieties. See Section 3.3 in [F]. (elaborate?) $\diamond$

2 Effective divisors

Definition 2.1. Effective divisors.

For a Weil divisor $D = \sum_i a_i D_i$ where D_i are prime Weil divisors, if all the nonzero $a_i > 0$, we call it an effective Weil divisor.

For a Cartier divisor on $X = \cup_i U_i$, if $f_i \in \mathcal{O}(U_i) \cap \mathcal{K}^*(U_i)$, then we say it is an effective divisor.

Check these two definitions are compatible when a Weil divisor is also a Cartier divisor.

Next in the relative situation $X \in \mathfrak{Sch}_S$ we will define relative effective Cartier divisors. Recall the definition of flat morhisms.

Definition 2.2. A morphism $f : X \to S$ with $f(x) = s$ is said to be flat at x if $f_s^\sharp : \mathcal{O}_{S,s} \to \mathcal{O}_{X,x}$ is a flat ring homomorphism. f is said to be a flat morphism if it is flat at every point $x \in X$.

Now suppose $f : X \to S$ is a flat morphism, $D \subset X$ is an effective Cartier divisor, we say it is also a relative divisor if the following two equivelant conditions are satisfied:

- $f|_D : D \to S$ is flat.
- $\forall x \in D$, f_x is not a zero divisor in the fibre $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} k_s$.

See Definition 1.11 from [JK]. Instead of proving the above two conditions are equivelant, we will simply state a stronger result (Lemma 9.4.3 from [FAG].)

Now D is still an effective Cartier divisor on X, if we don't require X is flat over S apriori, we still have:

Lemma 2.3. *TFAE:*

- *D is flat at x.*
- *X and D are flat at x, and the fiber D_s is an effective divisor in $X_s := X \otimes_{\mathcal{O}_{S,s}} k_s$.*
- *X is flat at x, and f_x is not a zero divisor in the fibre $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} k_s$.*

Example 2.4. $S = \mathbb{Z}$, $X = \mathbb{Z}[Y]$ (in particular is a UFD.) The divisor given by the height 1 prime ideal (p), where p is a prime number in $\mathbb{Z}$, is not relative effective. (Why?) The height 1 prime ideal $(f(Y))$ where $f(Y)$ is a monic irreducible polynomial gives a relative effective divisor. (Why?) Note however although we start from a Cartier divisor which is also a prime Weil divisor, the fibre $\overline{f(Y)}$ mod p doesn't have to be irreducible.

Now let $X \in \mathfrak{Sch}_S$ be flat over the base S, we define a functor $CDiv_{X/S}$ from $\mathfrak{Sch}_S$ to the category of Set:

Definition 2.5. $\forall T \in \mathfrak{Sch}_S$,
$CDiv_{X/S}(T) = \{$relative effective divisors on $X_T/T\}$.

Remark 2.6. That this is indeed a functor is essenatially due to the fact that tensor product is right exact and keeps flatness. (Why?) $\diamond$

Recall the definition of representable functors. A contravarient functor $\mathcal{F}$ from Cat $\mathcal{A}$ to Set is representable if there is a natural isomorphism from $\mathcal{F}$ to $h_B := Hom_{\mathcal{A}}(-, B)$. B is an object in $\mathcal{A}$ that represents the functor $\mathcal{F}$. This means there is an universal object $U \in \mathcal{F}(B)$, such that $\forall Z \in Cat\mathcal{A}$, $Hom_{\mathcal{A}}(Z, B) \to \mathcal{F}(Z)$ given by $g \to g^*U$ is an isomorphism.

Theorem 2.7. *Let S be locally noetherian, $X \in \mathfrak{Sch}_S$ be flat and projective. Then*

- *$CDiv_{X/S}$ is representable by an open subscheme of $Hilb_{X/S}$. (we will abuse notation and denote the scheme represents it also $CDiv_{X/S}$.)*
- *If X/S is smooth, then $CDiv_{X/S} \hookrightarrow Hilb_{X/S}$ is universally closed.*

See Theorem 1.13 in [JK].

(Theorem 1.4 in [JK]. Let X/S be a projective scheme, $\mathcal{O}(1)$ a relatively ample line bundle, P a polynomial. The functor $Hilb_P(X/S)$ is representable by $U \in X \times_S Hilb_p(X/S)$. $Hilb_P(X/S)$ is projective over S.)

3 Sheaf associated to a Cartier divisor

First recall several definition:

Sheaf of $\mathcal{O}_X$ module. $\mathcal{F}(U)$ is an $\mathcal{O}_X$ module for any open U, and for $V \subset U$, $(af)|_V = a|_V f|_V$.

Let $\mathcal{F}$ be an $\mathcal{O}_X$ module, $\mathcal{F}$ is quasi coherent if $\forall x \in X$, there is an open neigborhood U s.t. there is an exact sequence of $\mathcal{O}_X$ modules

$$\mathcal{O}_X^J|_U \to \mathcal{O}_X^I|_U \to \mathcal{F}|_U \to 0$$

Characterization: Let X be an scheme, and $\mathcal{F}$ an $\mathcal{O}_X$ module. Then $\mathcal{F}$ is quasi coherent iff $\forall$ affine open U, $\mathcal{F}(U)^\sim = \mathcal{F}|_U$. (Liu Theorem 5.1.7.)

For an arbitrary coherent sheaf on a Noetherian integral scheme, we can define its rank to be the rank of K vector space at the generic point.

Quasi coherent sheaves form a monoid under tensor product. The identity element being $\mathcal{O}_X$. We call the invertible elements invertible sheaves. These are the locally free sheaves of rank 1. (Exercise). For a locally free sheaf of rank 1 $\mathcal{G}$, its inverse is $\mathcal{G}^\vee := Hom_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X)$.

For a Cartier divisor D, represented by (U_i, f_i) as above. We can define the subsheaf of $\mathcal{K}$: for each U_i, form the sub $\mathcal{O}_X|_{U_i}$ module of $\mathcal{K}$ as $(f_i^{-1}\mathcal{O}_X(U_i))^\sim$. (check well defined.) This is locally free of rank 1 (and it is easy to see it is invertible.) Denoted by $\mathcal{L}(D)$.

$\mathcal{L}(D)$ can also be charaterized by $\mathcal{L}(D)(U) = \{f \in \mathcal{K}^*, div(f) + D|_U > 0\} \cup 0$. (Why?) (This definition easily generalize to the sheaf associated with a Weil divisor.)

Description: D is an effective Cartier/Weil divisor iff $\mathcal{O}_X \subset L(D)$ as subsheaves of K.

Theorem 3.1. *a) $D \to \mathcal{L}(D)$ gives a one-one correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$.*
b)

$$\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$$

c) $D_1 \sim D_2$ iff $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$ as abstract sheaves, regardless of embedding in $\mathcal{K}$.

Proof: a) One direction is obvious. For the other direction, D can be recovered from $\mathcal{L}(D)$ by taking f_i to be the inverse of a local generator of $\mathcal{L}(D)|_{U_i}$.

b)

$$\mathcal{L}(D_1 - D_2) = \mathcal{L}(D_1) \cdot \mathcal{L}(D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)$$

c) By b), it will suffice if $\mathcal{L}(D) \cong O_X$ iff D is principal. If D is principal, sending $1 \to f^{-1}$ gives $O_x \cong \mathcal{L}(D)$. Conversely also. Q.E.D.

Proposition 3.2. *When X is integral, every torsion free sheaf of rank 1 is isomorphic to a subsheaf of $\mathcal{K}$.*

In the following we assume X is Noetherian integral.

Definition 3.3. A coherent sheaf $\mathcal{F}$ is called reflexive if the natural map $\mathcal{F}^{\vee\vee}$ is an isomorphism.

Proposition 3.4. *1. $\mathcal{F} \hookrightarrow \mathcal{F}^{\vee\vee}$ is an embedding if $\mathcal{F}$ is torsion free.*
2. For an arbitrary coherent $\mathcal{F}$, $\mathcal{F}^{\vee}$ is always reflexive.

Theorem 3.5. *Let $\mathcal{L}$ be a reflexive subsheaf of $\mathcal{K}$ on a normal variety X, TFAE:*
a) $\mathcal{L}$ is rank 1.
b) $\mathcal{L} = \mathcal{L}(D)$ for some Weil divisor D.

The definitions and proof here from Chapter 8 in [CoX] and also [KS].

The next few propositions will help us characterize the relation between Weil divisors and Cartier divisors on a normal variety.

Proposition 3.6. *Let X be normal Noetherian, (not necessarily quasi projective), $\mathcal{F}$ coherent, then $\mathcal{F}$ is S_2 iff reflexive.*

Proposition 3.7. *Let X be Noetherian, $\mathcal{F}$ is S_2 coherent on X. $Y \subset X$ be a closed subset of $codim(Y/X) \geq 2$, $U := X \setminus Y$, then $\mathcal{F}(X) \to \mathcal{F}(U)$ is an isomorphism.*

Corollary 3.8. *X normal Noetherian, $\mathcal{F}$ is S_2, $\mathcal{F} \cong i_*\mathcal{F}|_U$.*

On the level of divisors, we have a Weil divisor D on X $\Rightarrow$ $D|_U$ is a Cartier divisor on U. For a Cartier divisor E on U, its Zariski closure $\overline{D}$ is a Weil divisor on X. We already see what happens on the level of their ssociated sheaves above. This is a one-one correspondence.

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