Here are some additional comments on my list of oral problems.

1. First a word on the Chinese Remainder theorem, let’s take the relation 
\[ p \equiv 7 \mod 8 \] and \[ p \equiv 1 \mod 5. \] If you want to find the (unique) congruence 
class \( mod \ 40, \) there is a procedure to do that rather than listing everything, the 
situation is a bit more general:

Take pairwise coprime numbers \( m_1, m_2, \cdots, m_n, \) define \( M_i := \Pi m_1 \cdots \hat{m}_i \cdots m_n, \)
and \( M_i^{-1} \) is any number that represents the inverse of \( M_i \mod m_i. \) Then the
solution of the system of congruence relations 
\[ p \equiv a_1 \mod m_1, \cdots, a_n \mod m_n \]
is equal to 
\[ \sum a_i M_i M_i^{-1} \mod \Pi m_i. \]

2. For the question of asking “for which primes \( p \) is 10 a fourth power modulo
\( p?\)”, of course, if you want to give an answer in terms of congruence relations, the
ultimate congruence relation should come from class field theory. First let’s just
see how much one can see with naked eyes. If \( p = 4k + 3, \) then if 10 is a square
it is automatically a fourth power. (Why?) But if 10 is not a square, -10 is. So
10 is a fourth power anyway. If \( p = 4k + 3, \) then any integer is a fourth power
modulo that prime. If \( p = 4k + 1, \) the situation is a bit more complicated.

The Galois group of the splitting field \( K = \mathbb{Q}(i, \sqrt[4]{10}) \) of \( x^4 - 10 = 0 \) over \( \mathbb{Q} \)
is the dihedral group of order 8. It is order 8 group, and not abelian, otherwise ev-
ery subfield is Galois, which we know is not true. So the only candidate we have
is the dihedral group \( D_4 \) or the quaternion group. It is a cyclic quartic extension
of \( \mathbb{Q}(i), \) but not cyclic over the other two quadratic subfield, thus it is not the
quaternion group. The four roots of \( x^4 - 10 = \pm \sqrt[4]{10}, i \sqrt[4]{10}, -\sqrt[4]{10}, -i \sqrt[4]{10}, \) call
them \( \{1, 2, 3, 4\} \). Given \( D_4 \) the presentation \( \{\tau, \sigma \tau^4, \sigma^2, \tau \sigma \sigma \}, \) we can identify
the action of \( D_4 \) on \( \{1, 2, 3, 4\} \). The four elements in \( D_4 \) that each of them do not
generate a normal subgroup of order 2 are the elements \( \sigma, \sigma \tau, \sigma \tau^2, \sigma \tau^3. \) Pick \( \tau \)
for example, say it fixes \( \{1, 3\}, \) then it acts like \( (2, 4). \) \( \sigma \) acts like \( (1, 2, 3, 4). \) (Of
course this assignment is not canonical, everything is up to conjugation. I just
use it to simplify the discussion.) Now you can see directly, the quadratic sub-
field \( \mathbb{Q}(i) \) is fixed by \( \{1, \tau, \tau^2, \tau^3\}; \) \( \mathbb{Q}(\sqrt[4]{10}) \) is fixed by \( \{1, \sigma, \tau^2, \sigma \tau^2\}; \) \( \mathbb{Q}(i\sqrt[4]{10}) \) is
fixed by \( \{1, \tau^2, \sigma \tau, \sigma \tau^3\}. \) These three subfields are governed by the biquadratic
field \( \mathbb{Q}(i, \sqrt[4]{10}) \) which is fixed by the commutator subgroup \( \{1, \tau^2\}. \) If you just
adjoin one (then automatically its minus) roots of \( x^4 - 10 \to \mathbb{Q}, \) you can say
that field is fixed by any one of the babies \( \sigma, \sigma \tau, \sigma \tau^2, \sigma \tau^3, \) i.e. you are looking
for primes in \( \mathbb{Z} \) whose decomposition group is either trivial or contained in the
order two subgroup generated by one of the above babies.

How much more information do you get than the naked eyes?

Remark*: If you care about the ramification index of 2 in \( \mathbb{Q}(i, \sqrt[4]{10}), \) you can
ask whether it ramifies in \( \mathbb{Q}_2(i, \sqrt{10})/\mathbb{Q}_2(i) \). Say I pick a uniformizer \( i - 1 =: \pi \) in \( \mathbb{Q}_2(i) \). Then \( i = \pi + 1, 2 = \pi^2(1 + \pi), 10 \equiv 1 + \pi + \pi^2 \mod \pi^5, \) and \( 10 \equiv 1 + \pi \mod \pi^5 + U_2^5 \). The Hilbert symbol of 10 pairs with representatives of units in \( \mathbb{Q}_2(i) \) don’t all give you 1, thus \( \mathbb{Q}_2(i, \sqrt{10})/\mathbb{Q}_2(i) \) ramifies.

You can also verify the situation by class field theory: the Norm subgroup of \( \mathbb{Q}_2(i, \sqrt{10})^* \) in \( \mathbb{Q}_2^* \) is contained in the norm subgroup of \( \mathbb{Q}_2(i)^* \) and \( \mathbb{Q}_2(\sqrt{10})^* \), and those are two different subgroups of index 2, so their intersection is at most index 4. Therefore the ramification index of their compositum is 4. This argument applies in more general situations.

The total ramification index of 2 in \( K \) is 8, for 5 it is 4. In particular, if you want to know the minimal modulus of \( K \) above \( \mathbb{Q}(i, \sqrt{10}) \), it is a product of primes above 2 and 5. So if you are going after the situation for primes of the form \( 4k + 1 \), you probably can not express it just in terms of modulus condition with rational integers. (If you do, please email me!)

3. On constructing \( S_3 \) extensions over \( \mathbb{Q}_p \) where \( p \not| 2 \). This uses the transfer map on class field theory. The transfer map deals with the situation when you have a tower of Galois extension: \( K \subset F \subset L \), where \( H := \text{Gal}(L/F) < \text{Gal}(L/K) =: G \). The situation is you want to go from \( G^{ab} \) to \( H^{ab} \), please check the article on wikipedia on the group theoretic recipe. In particular, if \( H = [G, G] \), the transfer map is trivial, a result due to Furtwangler. If \( G \) is abelian, then \( \text{Ver}(g) = (\bar{g})^{[G:H]} \) where \( \bar{g} \) is the image of \( g \) in \( H \). (Why?) On the ideal class group side, the map goes from using an ideal of \( \mathcal{O}_K \) to generate an ideal in \( \mathcal{O}_F \), which is perfectly legitimate. In the local field situation, you just take an element of \( K^* \) and view it as an element in \( F^* \).

Books by Gras/Artin and Tate on class field theory has statements about the transfer (Verlagerung) map.