“My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this”

Signature __________________________

This exam contains 8 pages (including this cover page) and 6 questions. Total of points is 120.

- Check your exam to make sure all 8 pages are present.
- You may use writing implements on both sides of a sheet of 5”x7” paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

<table>
<thead>
<tr>
<th>Question</th>
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<th>Score</th>
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<td>Total:</td>
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Boundary value problems:

\[
\phi''(x) = -\lambda \phi(x)
\]

### Boundary conditions

<table>
<thead>
<tr>
<th>Condition</th>
<th>(\phi(0) = 0)</th>
<th>(\phi(L) = 0)</th>
<th>(\frac{d\phi}{dx}(0) = 0)</th>
<th>(\frac{d\phi}{dx}(L) = 0)</th>
<th>(\phi(-L) = \phi(L))</th>
<th>(\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L))</th>
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### Eigenvalues \(\lambda_n\)

\[
\left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \ldots
\]

### Orthogonality

\[
\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \begin{cases} 
0, & n \neq m \\
L/2, & n = m \neq 0 
\end{cases}
\]

\[
\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = \begin{cases} 
0, & n \neq m \\
L/2, & n = m \neq 0 \\
L, & n = m = 0 
\end{cases}
\]

\[
\int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \begin{cases} 
0, & n \neq m \\
L, & n = m \neq 0 \\
L/2, & n = m = 0 
\end{cases}
\]

\[
\int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = \begin{cases} 
0, & n \neq m \\
L, & n = m \neq 0 \\
2L, & n = m = 0 
\end{cases}
\]

\[
\int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = 0
\]

Orthogonality
1. (20 points) 1. Compute the Fourier cosine series for the function \( f(x) = x^2 \) on the interval \([0, \pi]\). Fully simplify your answer. Hint:

\[
\int x^2 \cos(nx) \, dx = \frac{2nx \cos(nx) + (-2 + n^2x^2) \sin(nx)}{n^3}
\]

2. Does the Fourier cosine series converge to the function \( f \) at the point \( x = 0 \)? Justify your answer.

3. Sketch the values of the Fourier cosine series of \( f \) on the interval \([-\pi, 2\pi]\), marking any points of discontinuity.

4. Demonstrate from your calculations that

\[
1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.
\]
2. (20 points) The displacement of a string $u(x,t)$, $0 \leq x \leq L$, $t \geq 0$ satisfies the following wave equation

$$u_{tt} = 8u_{xx} - u$$

with boundary conditions $u(0, t) = 0$, $u(L, t) = 0$ and initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. Find the solution $u(x, t)$. 
3. (20 points) Consider the boundary value problem

\[ \frac{x^2}{2} \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} + \lambda \phi = 0, \quad 1 \leq x \leq e \]

and

\[ \phi(1) = \phi(e) = 0. \]

Find the eigenvalues and eigenfunctions.
4. (20 points) 1. Write the following boundary value problem of $\phi(x)$

$$x^2 \phi'' + 2x \phi' + \lambda \phi = 0, \quad 1 \leq x \leq 3, \quad \phi(1) = \phi(3) = 0$$

in the standard Sturm–Liouville form.

2. Is this a regular eigenvalue problem? Why?

3. Show that all the eigenvalue $\lambda > 0$. 

5. (20 points) Solve the 2D wave equation on rectangle $\Omega = [0, 4] \times [0, 9]$

\[ u_{tt} = c^2 \Delta u \]

with boundary conditions $u(x, y, t) = 0$ on the boundary $\partial \Omega$ and initial conditions $u(x, y, 0) = \sin(\pi x) \sin(\pi y)$ and $u_t(x, y, 0) = 0$. 

6. (20 points) Consider the heat equation of the temperature \( u(x,y,t) \) on a bounded 2D region \( \Omega \)

\[
    u_t = \Delta u - u, \ (x,y) \in \Omega, \ t \geq 0
\]

with boundary condition \( u|_{\partial \Omega} = 0 \)

1. Let

\[
    E(t) = \int_\Omega (u(x,y,t))^2 \, dx\,dy.
\]

Show that the function \( E(t) \) is nonincreasing, in other words

\[
    \frac{d}{dt} E(t) \leq 0.
\]

2. Use the function \( E(t) \) to show that the solution with fixed initial condition \( u(x,y,0) = f(x,y) \) is unique.