“My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this”

Signature ____________________________

This exam contains 11 pages (including this cover page) and 10 questions. Total of points is 100.

- Check your exam to make sure all 11 pages are present.
- You may use writing implements and a single handwritten sheet of 8.5”x11” paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

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1. (10 points) State the definition of an operation of group $G$ on a set $S$. State the property for the operation to be transitive.

**Definition of operation:**

A map $G \times S \rightarrow S$

$(g, s) \rightarrow gs$

such that

1. $(g_1g_2)s = g_1(g_2s)$

2. $1s = s$, $\forall s \in S$

Transitive means the operation has only one orbit.
2. (10 points) Write the element $(123)(234) \in S_4$ as product of disjoint cycles.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 3 & 4 & 2 \\
2 & 1 & 4 & 3 \\
\end{array}
\]

\[
\text{So} \quad (123)(234) = (12)(34)
\]
3. (10 points) Find the Sylow 2-subgroup of $S_4$.

$$|S_4| = 4 \times 3 \times 2 \times 1 = 8 \times 3$$

Sylow 2-subgroups are subgroups of order 8.

Number of Sylow 2-subgroups $s = 1$ or 3.

Since every transposition $(i, j)$ is contained in some Sylow 2-subgroup and all the $(i, j)$ generate $S_4$, $s = 1$, $s = 3$.

Consider the action of $D_4$ on the vertices of a square. $D_4 \rightarrow S_4$ as a subgroup of $S_8$.

Consider the conjugation of $D_4$ in $S_4$ (or re-index the vertices).

$$H_1 = D_4 = \langle 1, (1234), (13)(24), (1432), (12)(34), (13), (24), (15)(23) \rangle$$

$$H_2 = \langle 1, (1324), (12)(34), (1423), (13)(24), (12), (13), (14)(123) \rangle$$

$$H_3 = \langle 1, (1342), (14)(23), (1234), (13)(124), (14), (23), (12)(134) \rangle$$
4. (10 points) Find all the normal subgroups of $D_6$.

$D_6 = \{ 1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y \}$

The conjugacy classes of $D_6$ are:

$\begin{array}{cccc}
1 & x^3 & x^6 & \frac{1}{3} x^4 \\
2 & x & x^2 & x^5 \\
3 & y & x^2y & x^3y \\
\end{array}$

$|D_6| = 12$, the normal subgroup is the union of conjugacy classes containing $\frac{1}{3}y$ whose order divides 12.

So

1. $\frac{1}{3}y$
2. $\frac{1}{3}x$, $\frac{1}{3}x^3$

Not a subgroup

3. $\frac{1}{3}x^2$, $\frac{1}{3}x^4$

Not subgroups
1+2+3, \{1, x^2, x^3, y, x^2y, x^3y\}

1+1+2 \{1, x, x^5, x^3y\} \{1, x^2, x^3, x^5y\}

Not subgroups.

1+1+2+3+3, D_4

6 normal subgroups.

\{1, x\}, D_4

\{1, x^3y\}, \{1, x^2, x^4y\}

\{1, x, x^2, x^3, y, x^2y\}

\{1, x^3, x^6, y, x^6y, x^6y\}
5. (10 points) Classify all finite groups of order 45.

Let $H$ be the unique Sylow 3-subgroup.

Let $K$ be the unique Sylow 5-subgroup.

Then $H$, $K$ are normal subgroups of $G$.

$|H \cap K| = 1$, $|H| = 3$,

$|H| = 9$, so $H \cong C_3 \times C_3$, or $C_9$.

$G \cong C_3 \times C_3 \times C_5$ or $C_9 \times C_5$. 
6. (10 points) Classify all finite groups of order 10.

\[ |G| = 10, \]

**number of Sylow 5-subgroup is 1.**

Let \( H \) be the Sylow 5-subgroup.

\( H \) is a normal subgroup of \( G \).

Let \( K \) be a Sylow 2-subgroup.

Then \( HK = G \), \( H \cap K = 1 \).

\[ G \cong H \times K, \quad \text{with} \]

\[ \varphi : \langle k \rangle \rightarrow \text{Aut}(H), \quad k \rightarrow \varphi(k) : h \rightarrow khk^{-1}. \]

Let \( 1 < \langle x \rangle, \quad x^5 = 1 \).

\( K = \langle y \rangle, \quad y^2 = 1 \).

\[ x^y = x^y, \quad \text{then} \quad y^2 \equiv 1 \pmod{5}. \]

So \( y \equiv 1 \pmod{5} \) or \( 4 \pmod{5} \).

\[ x^y = x, \quad 2y \equiv x^5 \equiv 1, \quad 2y \equiv 1 \pmod{5} \]

\[ G \cong \langle 2 \rangle \times S_5 \quad \text{or} \quad D_5. \]
7. (10 points) Prove that a group of order 200 is not a simple group.

\[ |G| = 200 = 2^3 \times 5^2 \]

**Number of Sylow 5-group**

\[ s \mid 8, \quad s \equiv 1 \pmod{5} \]

So \( s = 1 \).

So Sylow 5-group is unique, and it is a nontrivial normal subgroup.

So \( G \) is not simple.
8. (10 points) Prove that $SO(2)$ is isomorphic to $\mathbb{R}/\mathbb{Z}$.

\[
\phi : \mathbb{R} \rightarrow SO(2)
\]

\[
\theta \rightarrow \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

is a group homomorphism.

\[
\ker \phi = \mathbb{Z}
\]

So $SO(2) \cong \mathbb{R}/\mathbb{Z}$.
9. (10 points) The operation of finite group $G$ on set $S$ is transitive and $H$ is a normal subgroup of $G$. Prove that the orbits under the operation of $H$ on $S$ have the same number of elements.

Let $O_1$, $O_2$ be two orbits of the operation of $H$ on $S$.

Let $x \in O_1$, $y \in O_2$, then $y = gx$ for some $g \in G$ (because $G$ operation is transitive).

The stabilizer of $x$ under $H$ action is $H_x = H \cap G_x = \{ g \in H \mid gx = x \}$. $H_y = H \cap G_y$, $G_y = gGxg^{-1}$

So $H \cap G_y = (gHg^{-1}) \cap gGxg^{-1} = g(H \cap G_x)g^{-1} = gH_xg^{-1}$

So $|Hx| = |Hy|$, $|O_1| = \frac{|Hx|}{|H|} = \frac{|H|}{|H_y|} = |O_2|$
10. (10 points) Let \( p \) be a prime number. Prove the center \( Z(G) \) of a nonabelian group \( G \) of order \( p^3 \) must have order \( p \).

\[ G \text{ is a } p\text{-group. So } Z(G) \text{ is non trivial} \]

\[ G/Z(G) \text{ is also a } p\text{-group.} \]

Since \( G \) is nonabelian, \( Z(G) \neq 1 \).

\[ \text{so } |Z(G)| = p \text{ or } p^2. \]

If \( |Z(G)| = p^2 \), then \( |G/Z(G)| = p \)

\[ G/Z(G) \text{ is a cyclic group} \]

Let \( G/Z(G) = \langle xZ(G) \rangle \)

then any element in \( G/Z(G) \) has the form \( x^i y\), \( y \in Z(G) \)

\[ (x^i y_1)(x^i y_2) = x^{2i} y_1 y_2 = (x^i y_2)(x^i y_1) \]

\[ \forall y_1, y_2 \in Z(G), \text{ so } G \text{ is abelian} \]

Contradiction! So \( |Z(G)| = p \).