Math 371
Spring 2020
Midterm 1
2/20/2020
Time Limit: 80 Minutes

Name: ______________________

ID ______________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ______________________

This exam contains 9 pages (including this cover page) and 6 questions.
Total of points is 70.

- Check your exam to make sure all 9 pages are present.
- You may use writing implements on both sides of a sheet of 8"x11" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	20	
Total:	70	

1. (10 points) Define a symmetric bilinear form on $\mathbb{R}^3$ by $\langle X, Y\rangle = X^T A Y$ where $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find a basis v_1, v_2, v_3 such that $\langle v_i, v_j\rangle = 0$ for all $i \neq j$.

2. (10 points) Let $i = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}$ and $j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ be two elements in $SU(2)$. Determine whether they are in the same conjugacy class. If they are, find $P \in SU(2)$ such that $PiP^{-1} = j$. If not, state the reason.

3. (10 points) (a) Find an injective group homomorphism from $SO(2)$ to $SO(3)$.

   (b) Find an injective group homomorphism from $O(2)$ to $SO(3)$.

4. (10 points) Let $e_1 \cdots e_n$ be the standard basis of $\mathbb{C}^n$. Define the linear operation ρ of permutation group S_n on $\mathbb{C}^n$ by $\sigma \cdot e_i = e_{\sigma(i)}$. Denote by χ_ρ the corresponding character. Prove that $\chi_\rho(\sigma)$ is equal to the number of elements fixed by σ, i.e.

$$\chi_\rho(\sigma) = \text{the number of elements in } \{i \in \{1, 2, \cdots, n\} | \sigma(i) = i\}.$$

5. (10 points) Prove that any rotation in $SO(2)$ can be written as the product of two reflections in $O(2)$.

6. (20 points) Let W be the space of real trace-zero 2×2 matrices $W = \{A \in M_{2\times 2}(\mathbb{R}) | trace(A) = 0\}$. W has a basis $\mathbf{B} = (w_1, w_2, w_3)$, where

$$w_1 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(a) Show that the symmetric bilinear form defined by $\langle A, A' \rangle = \text{trace}\,(AA')$ has signature $(2, 1)$. (Hint: use basis $\mathbf{B}$)

(b) Prove that $P \star A = PAP^{-1}$ defines a linear group operation of $SL(2, \mathbb{R})$ on the space W.

(c) Use this operation to define a group homomorphism $\varphi : SL(2, \mathbb{R}) \to O_{2,1}$.

(d) Prove the kernel of this homomorphism is $\{\pm I\}$.

Draft 1:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

Draft 2:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.