SAMPLE SOLUTIONS HW 1

HW 1, Problem 8.4.10

Let V denote the vector space of real $n \times n$ matrices. Then $\langle A, B\rangle = trace(A^tB)$ defines a positive definite bilinear form on V, and find an orthonormal basis for this form.

Proof. First, we check bilinearity. Given matrices A_1, A_2, we have

$$\langle A_1+A_2, B\rangle = \text{trace}((A_1+A_2)^tB) = \text{trace}((A_1^t+A_2^t)B) = \text{trace}(A_1^tB)+\text{trace}(A_2^tB) = \langle A_1, B\rangle+\langle A_2, B\rangle,$$

and for $\lambda \in \mathbb{R}$ and a matrix A, we have

$$\langle \lambda A, B\rangle = \text{trace}((\lambda A)^tB) = \text{trace}(\lambda A^tB) = \lambda\text{trace}(A^tB) = \lambda\langle A, B\rangle.$$

We use the fact that trace : $M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is a linear map, and some basic properties of transposes. This shows linearity in the first component. Similarly, one shows linearity in the second component.

For $A = (a_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{R})$, we have that $A^tA = (b_{ij})_{1\leq i,j\leq n}$, where $b_{kk} = \sum_{i=1}^n a_{ik}^2$. Hence $\text{trace}(A^tA) = \sum_{k=1}^n\sum_{i=1}^n a_{ik}^2$. From this it follows that $\langle A, A\rangle \geq 0$ for all $A \in M_n(\mathbb{R})$, and $\langle A, A\rangle = 0$ if and only if $A = 0$, showing positive definiteness of the given form.

Next we find an orthonormal basis. Note that $\dim_{\mathbb{R}} M_n(\mathbb{R}) = n^2$ so our basis will have n^2 elements. A natural guess for a basis for $M_n(\mathbb{R})$ is the collection $\{e_{i,j}\}_{1\leq i,j\leq n}$ of matrices, where $e_{i,j}$ has 1 in the (i, j)th location, and 0 elsewhere. If one identifies $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$ (as $\mathbb{R}$-vector spaces), then these $e_{i,j}$'s correspond exactly to the standard basis elements of $\mathbb{R}^{n^2}$. Now, given our formula above, one easily computes that $\langle e_{i,j}, e_{i,j}\rangle = 1$, and $\langle e_{i,j}, e_{k,l}\rangle = 0$ if $(i, j) \neq (k, l)$. Hence, the above collection yields an orthonormal basis. □

HW 1, Problem 8.4.11

Let W_1, W_2 be subspaces of a vector space V with a symmetric bilinear form. Prove that $(a)(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$, $(b)W \subseteq W^{\perp\perp}$, $(c)W_1 \subseteq W_2 \implies W_2^\perp \subseteq W_1^\perp$.

Proof. We first prove (c). Take $x \in W_2^\perp$. Then $\langle w_2, x\rangle = 0$ for all $w_2 \in W_2$ (by definiton). In particular, $\langle w_1, x\rangle = 0$ for all $w_1 \in W_1$, since $W_1 \subseteq W_2$. Hence $x \in W_1^\perp$, and $W_2^\perp \subseteq W_1^\perp$, as required.

For (a), we note that $W_i \subseteq W_1 + W_2$ for $i = 1, 2$. By (c), $(W_1 + W_2)^\perp \subseteq W_i^\perp$ for $i = 1, 2$, and hence $(W_1 + W_2)^\perp \subseteq W_1^\perp \cap W_2^\perp$. For the reverse containment, take any $x \in W_1^\perp \cap W_2^\perp$, and any $z \in W_1 + W_2$. We have $z = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$, and $\langle w_1 + w_2, z\rangle = \langle w_1, z\rangle + \langle w_2, z\rangle = 0$, whence $x \in (W_1 + W_2)^\perp$, and we have $W_1^\perp \cap W_2^\perp \subseteq (W_1 + W_2)^\perp$, proving (a).

Finally, for (b), we take any $w \in W$, and any $x \in W^\perp$. Then $\langle w, x\rangle = 0$, whence $w \in W^{\perp\perp}$. Consequently, $W \subseteq W^{\perp\perp}$, as required. □

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