180 Options (Nov. 23)

Traveling to Europe

You are traveling to Europe in one month, and you need €1000 before traveling.

The current rate is €1 = $1

You only have $40 now.

By the end of the month, you'll get another $1000.

Should you wait for a month to buy €?
The problem is the currency exchanging rate is always changing.
Assume the rate for € will change 5% (±) monthly.
If the rate drops to 0.95
\[ \text{€} 1 = \text{US} 0.95 \]
You need $950 < $1040.
You are good.
If the rate increases to 1.05
\[ \text{€} 1 = \text{US} 1.05 \]
You need $1050 > $1040
You are in trouble 😞
One possibility: borrow money $960 from the bank. Suppose the monthly interest rate is 1%, compounded monthly.

$960 + $40 \implies 1000$

After one month pay the bank back

$960 \times (1 + 1\%) = 969.6 < 1000$

Good.

But bank is unwilling to lend you money!
Now comes along an option trader.

An option trader offers an option: 期权

- In one month you have the option to buy €1,000 for $1,000 (You may or may not exercise it)

- Buying the option costs you $29.70  

29.70 < 40 good

We will talk about how to compute this
If the rate \( \downarrow \) to \( €1 = $0.95 \)
You don't exercise the option
Instead buy €1600 for
$950 from the market.
Total cost $950 + 29.70

If the rate \( \uparrow \) to \( €1 = $1.05 \)
You exercise the option,
buy €1000 from the trader
for $1000
Total cost $1000 + 29.70
There is also a type of contract where you agree to buy $1,000 for $4,000, regardless of whether the rate drops or not.

This is future 期货.

Our basic question is how to price an option. So from now on imagine yourself as a trader, not the buyer anymore.

As a trade: 1) Decide the option price 2) Make a portfolio.
The goal is to make the cash flow zero. No profit No loss

Portfolio is

1) buying € for $x

2) borrowing money $y from the bank (This is the advantage of a trader)

Let's see how to achieve the goal in this exam.

Suppose you have a portfolio now

(Use $x to buy €x

( borrow $y from the bank )
After one month.

If rate > 1.05

The holder of the option exercise it. He will use $1000 to buy €1000 (which will be worth $1050) from you. So you will lose $50 = 1000 \times (1.05 - 1)

In order to make the cash flow 0, you need to make $50 profit from your portfolio.

You need the portfolio to be worth $50 (to make up the loss)
If the rate is 0.95

The buyer won't exercise the option. In order to make the cash value 0. You need the portfolio to be worth 0.

In summary, after one month

The portfolio is

\[ $50 \text{ if } \varepsilon 1 = $1.05 \]

\[ < 0 \text{ if } \varepsilon 1 = $0.95. \]

How to achieve this?
If \( P = \) option price is \$29.70

Borrow \( y = 470.30 \) from the bank.

So you have \( x = P + y \)

\[ = \$500 \]

Buy \( \€500 \) for \$500.

( Cash flow = 0).

If \( \€1 = \$1.05 \) Your portfolio becomes

\[ 500 \times 1.05 - 470.30 \times 1.05 = 50 \]

The value of \( \€500 \) due to

the bank
If $1 = \$0.95$

Your portfolio is worth

$$500 \times 0.95 - 470.30 \times 1.01 = 0$$

↑

value of €500 over to the bank.

This is a very convenient point of view. We just focus on the portfolio. What the trader actually does is more complicated.

If $1 = \$0.95$

You have €500.

You sell €500 for $470.30 and return it to the bank.
If 1 = $1.05

The buyer wants to buy €1,000 for $1,000 from you.

You take $1,000 from the buyer. 1,000 = 525 + 427.30.

You buy €500 for $525 plus the €500 already in your portfolio

⇒ give €1,000 to the buyer

Also, pay $427.30 to the bank.
We just focus on the portfolio, ignore other trading. This is because other trading are trading. They will always add up your portfolio.

Also the initial portfolio. We regard the value of $E$ as $+$, the money borrowed from the bank as $-$. This is different from the cash flow. As cash flow you get $y$ from the bank, you pay $x$ for the $E$. 
The price of the option 

\[ P = \text{the value of your initial portfolio} \]

\[ = x - y. \]

You can understand it in two ways. ① The buy pays P for the portfolio.

② The cash flow for the trader. borrow y from the bank 
get P from the buyer.

Pay x for €

\[ y + p - x = 0 \leftarrow \text{cash flow is 0.} \]
Nov 25. Example I

Now we suppose (continue)

\[ P = \text{Option price} \]
\[ = \text{initial worth of the portfolio.} \]

We use the binomial tree
(similar to the decision tree)

\[ P_u = \text{The portfolio if } \varepsilon \in U \]

\[ P_d = \text{The value of portfolio if } \varepsilon \in \bar{U} \]

\[ P_u \varepsilon_1 = 1.05 \]

\[ P \leftarrow \]
\[ P_d \varepsilon_1 = 0.95 \]
We know

\[
\begin{align*}
P_u &= 1.05x - 1.01y = 50 \\
P_d &= 0.95x - 1.01y = 0
\end{align*}
\]

\[\Rightarrow\] solve for \(x, y\)

\[
\begin{align*}
x &= 500 \\
y &= 470.30
\end{align*}
\]

\[\Rightarrow\] \(P = x - y = 29.70\)

There is another way

Rewrite

\[
\begin{align*}
P_u \Rightarrow (1.05 - 1.01)x + 1.01(x-y) \\
P_d \Rightarrow (0.95 - 1.01)x + 1.01(x-y)
\end{align*}
\]
Now you want to eliminate $X$.

multiply $P_a$ by $p$

and $P_d$ by $(1-p)$

So that

$$p(1.05 - 1.01)$$

$$= (1-p)(1.01 - 0.95)$$

How to understand $p$?

Key idea: The portfolio doesn't care about the probability that the exchange rates go up or down.
Choose a probability \( p \) so that \( x \) can be eliminated in other words, the expected value of the portfolio is independent of how much \( x \) is.

**Risk - Neutral Pricing:**

Expected value of your investment = Expected value of money in the bank.

In our case, \[ \text{Expected value of } \mathcal{E} = \text{the money in the bank}. \]
$1 in the bank
after one month
\[1 \times (1 + 1\%) = 1.01\]
$1 \rightarrow 1$

Suppose probability \( p \)

rate \( r \)

probability \( 1 - p \) the rate \( r \)

Expected value =
\[ p \times 1.05 + (1 - p) \times 0.95 \]

Want
\[ p \times 1.05 + (1 - p) \times 0.95 = 1.01 \]
\[ p = 0.6 \implies \text{risk neutral} \]

Now the expected value of the portfolio after one month:

\[ p \cdot P_n + (1-p) \cdot P_d \]

\[ = 50 \times 0.6 + 0 \times 0.4 \]

\[ = 30. \]

The present value is:

\[ P = \frac{30}{1.01} = 29.70 \]

This is \text{risk-neutral pricing}.
Another example II

Suppose currently $1 = \frac{1.05}{0.95}$

After one month

\[
\frac{1.05}{0.95} \leq \left( \frac{1.05}{0.95} \right) 1.05
\]

How much will an option to buy €1000 for $1000 cost?

What is the portfolio?

If rate / $P_u = \left( \frac{(1.05)^2}{0.95} - 1 \right) \times 1000$

$= 160.53$
If rate \( \nu \) then \( P_d = (1.05 - 1)x/1000 = 50 \)

\( p \) is the same
\( 0.05p + 0.95(1-p) = 1.0 \)

**EV of Portfolio**
\[
P_1 = 0.6P_u + 0.4P_d
\]

Optim price = initial value of portfolio is
\[
P = \frac{P_1}{1.01} = \frac{0.6P_u + 0.4P_d}{1.01} = 115.17
\]

\[
1.05x - 1.01y = P_u \quad (1)
\]
\[
0.95x - 1.01y = P_d \quad (2)
\]
\[(1 - 0.95) \times X = P_n - P_d\]
\[X = \frac{160.53 - 50}{0.1}\]
\[= 1,105.3\]
\[y = x - P\]
\[= 1,105.3 - 115.17\]
\[= 990.13\]

2 months

Suppose the option is to buy €1,000 for $1,000 in two months.

Current rate £1 = $ \frac{1}{0.95}
Option trader will adjust the portfolio at the end of one month.

The same question.

Option Price P?
Portfolio x, y?
After one month how to adjust if rate r?
\[
\frac{1.05}{0.95} \left( \frac{1.05}{0.95} \right)^2
\]

\[
\frac{1}{0.95}
\]

1st month

2nd month
The idea is

1) Focus on the Portfolio

2) Work back

\[ P < P_u < P_{nn} \]
\[ P_d < P_{nd} \]
\[ P_{dn} \]
\[ P_{dd} \]

After two months, the two cases are previous examples.

\[ P_{nn} = \left( \frac{(1.05)^2}{0.95} - 1 \right) \times 1000 \]

\[ = 160.53 \]
\[ P_{ud} = P_{du} = (1.05 - 1) \times 1000 = 50 \]

\[ P_{ud} = 0 \]

**Work back**

We know

\[ P_d = \left( \text{The option price in Example I} \right) = 29.70 \]

\[ P_u = \left( \text{The option price in Example II} \right) = 115.17 \]
p is the same = 0.6.

\[ 1.01P = pP_u + (1-p)P_d \]

We can compute it directly from \( P_u, P_d, \) and \( P_d. \)

\[ (1.01)^2 P = p^2 P_u + p(1-p)P_d \]
\[ + (1-p)P_d u \]
\[ + (1-p)^2 P_d d \]

\[ P = \frac{(0.6)^2 \times 160.53 + 0.48 \times 56 + 0}{(1.01)^2} \]
\[ = 80.18 \]
For \( x \)

\[
(1.05 - 0.95) x = P_u - P_d \\
= 115.17 - 29.70
\]

\[
x = \frac{115.17 - 29.70}{1.05 - 0.95}
\]

\[
= 854.70
\]

\[
y = x - P = 854.70 - 80.18 \\
= 774.52
\]

After one month, if the rate go up \( \uparrow \). Adjust to the portfolio in Example II.