Apr. 12

5.11 a cycle of generalized eigenvectors.
A $n \times n$

We want a basis of generalized eigenvectors in a special form.
Definition

\( \tau \) a generalized eigenvector of \( A \) corresponding to \( \lambda \).

If \( p \) the smallest positive integer such that \( (A - \lambda I)^{p-1} \tau = 0 \),

Then the ordered set

\[
\{(A - \lambda I)^{p-1} \tau, (A - \lambda I)^{p-2} \tau, \ldots, (A - \lambda I) \tau, \tau\}
\]

is called a cycle of generalized eigenvectors of length \( p \).
\begin{align*}
\overline{\mathbf{v}}_1 &= (A - \lambda \mathbf{I})^{p-1} \overline{\mathbf{v}} \\
\overline{\mathbf{v}}_2 &= (A - \lambda \mathbf{I})^{p-2} \overline{\mathbf{v}} \\
\vdots \\
\overline{\mathbf{v}}_i &= (A - \lambda \mathbf{I})^{p-i} \overline{\mathbf{v}} \\
\vdots \\
\overline{\mathbf{v}}_p &= \overline{\mathbf{v}} \\
\overline{\mathbf{v}}_{i-1} &= (A - \lambda \mathbf{I}) \overline{\mathbf{v}}_i
\end{align*}
Important for 2 reasons

1) A cycle of generalized eigenvectors is L.I.

2) \[ \begin{array}{cccc}
\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \\
\end{array} \]

Since \( \vec{v}_{i-1} = (A - \lambda I) \vec{v}_i \) for \( i > 1 \)

we have

\[ A \vec{v}_i = \lambda \vec{v}_i \]

\[ A \vec{v}_i = \lambda \vec{v}_i + \vec{v}_{i-1} \ (i > 1) \]

Without this, it would be an \( \text{eigen-} \) vector
Theorem
A cycle of generalized eigenvectors is L.I.

Proof:
Assume
\[ c_1 \vec{v} + c_2 (A - \lambda I) \vec{v} + \ldots + c_p (A - \lambda I)^{p-1} \vec{v} = \vec{0} \]

Want to show all \( c_i \)'s = 0

Apply \( (A - \lambda I)^{p-1} \)
all the terms become zero except
\[ c_1 (A - \lambda I)^{p-1} \vec{v} = \vec{0} \]
\( \neq \vec{0} \)
Thus \( c_1 = 0 \)

Then apply \((A - \lambda I)^{p-2}\)

\[\Rightarrow c_2 (A - \lambda I)^{p-1} \frac{1}{v} = 0\]

Thus \( c_2 = 0 \)

Continue this way,

until \( c_p = 0 \) \[\square\]
The consequence is for any $A_{n \times n}$, we can find a basis $\beta$ consisting of cycles of generalized eigenvectors of $A$, and relative to this basis $\beta$

$J = (A)^{\beta} \beta$ is of a special form. Each column is like

$[A \bar{u}_i]_{\beta} = (\lambda \bar{u}_i + \bar{u}_{i-1}) \bar{\beta}$

$= \begin{pmatrix} 0 \\ \lambda \\ 0 \end{pmatrix}_{-1}$
A Jordan block of size $r$ corresponding to $\lambda$ is

$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \\ & & & \ddots \\ & & & & \lambda \end{bmatrix}_{r \times r}$$

$r=1 \quad r=2 \quad r=3$

$$[\lambda] \quad [\lambda 1] \quad [\lambda 1 0]$$

$$[0 \lambda] \quad [0 \lambda 1] \quad [0 0 \lambda]$$
Definition

A square matrix is said (JCF) to be in Jordan canonical form if it consists of Jordan blocks along the diagonal.

In particular if all Jordan blocks have size 1, this is a diagonal matrix.
Nondefective
similar to
D diagonal
has a
basis of eigenvectors
\{ \overline{v}_1, \ldots, \overline{v}_n \}
\quad S = [ \overline{v}_1 \ldots \overline{v}_n ]
\quad S^{-1}AS = D

General
similar to
J Jordan form
has a basis of cycles of generalized eigenvectors
\{ \overline{u}_1, \ldots, \overline{u}_n \}
\quad S = [ \overline{u}_1, \ldots, \overline{u}_n ]
\quad S^{-1}AS = J
Given $A$ how to get $S$ and $J$?

# cycle corresponding to $\lambda$

$=$ # Jordan blocks corresponding to $\lambda$

$=$ g.m. of $\lambda$

Total size of Jordan blocks corresponding to $\lambda = \text{a.m. for } \lambda$

Two approaches

bottom up (only 1 Jordan block corresponding to $\lambda$)

top down (general)
Example (bottom up)

\[ A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 2 \end{bmatrix} \]

\[ p(x) = (x-2)^3 \quad \lambda = 2 \]
\[ \alpha.m. = 3 \]

\[ A - \lambda I = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \]

rank = multiplicity = 1

eigenvector \[ \bar{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

g.m=1 only one Jordan block
$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

has a cycle of length 3

$\begin{array}{c|c|c} \overline{u}_1 & \overline{u}_2 & \overline{u}_3 \\ \hline \text{eigenvector} & (A - \lambda I)^{p-1} & \overline{v} \end{array}$

$\overline{u}_{i-1} = (A - \lambda I) \overline{u}_i$

Knowing $\overline{u}_{i-1}$, solve for $\overline{u}_i$
\[ \overline{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 \\
3 & 2 & 0 & 0 & 1
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ \overline{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \]

\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
-1 & -1 & 0 & -1 \\
3 & 2 & 0 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ \overline{v}_3 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} \]

A cycle is \[ \left \{ \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} \right \} \]