Midterm #1 for Math 240, Spring 2017

Name (printed): ____________________________

My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this midterm examination.

Signature: ____________________________ Date: ____________________________

Recitation TA: ____________________________ Recitation Hours: ____________________________

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<th>Problem</th>
<th>Points</th>
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- You have 80 minutes for this exam.
- Please show ALL your work on this exam paper. Partial credit will be awarded where appropriate. Answers with little or no justification will get no credit.
- You may use both sides of one 8.5 by 11 inch sheet of notes.
- NO books, laptops, cell phones, calculators, or any other electronic devices may be used during the exam.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.
- Box your final answers.
1. (10 points) Find the inverse of the following $3 \times 3$ matrix:

$$C = \begin{pmatrix}
1 & 2 & 1 \\
0 & -1 & 3 \\
2 & 3 & 4
\end{pmatrix}$$
2. Solve the system of linear equations, in variables \( x_1, x_2, x_3 \) and \( x_4 \), whose augmented matrix is given below:

\[
\begin{pmatrix}
1 & -2 & 2 & -1 & | & 3 \\
3 & 1 & 6 & 11 & | & 16 \\
2 & -1 & 4 & 4 & | & 9
\end{pmatrix}.
\]
3. Are the following three statements, given below, true or false? If the statement is true, justify your answer completely. If the statement is false, provide an explicit counterexample explaining why it is false. Answers with no or incorrect justification will receive no credit.

(a) (4 points) If \( CD = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) for two \( 2 \times 2 \) matrices \( C \) and \( D \), then the product \( DC \) also equals \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).

(b) (3 points) The matrix \( \begin{pmatrix} x^2 & x \\ y^2 & y \end{pmatrix} \) is not invertible if and only if \( x = 0 \) or \( y = 0 \).

(c) (3 points) The set of all \( 2 \times 2 \) matrices \( A \), with \( \det(A) < 0 \), is a subspace of \( M_2(\mathbb{R}) \).
4. (10 points) Let $W$ be the set of polynomials $p(x)$ (in one variable $x$) of degree less than or equal to four with the property that $p(1) = 0$. Prove that $W$ is a vector space.
5. (2 points) Let $U = \begin{pmatrix} 10 & 0 \\ 0 & -1 \end{pmatrix}$. Let $M = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$. Let $N = MUM^{-1}$. Find $N^{10}$. 
6. (10 points) Let \( D = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} \), \( \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \) and \( E = \begin{pmatrix} 1 & 2 & 0 & b_1 \\ 2 & 0 & 2 & b_2 \\ 2 & 1 & 1 & b_3 \end{pmatrix} \).

(a) (6 points) Compute the row reduced echelon form of \( E \).

(b) (4 points) Obtain the algebraic equation(s) that describe the vector space spanned by the column vectors of the matrix \( D \) in \( \mathbb{R}^3 \).
7. (a) (6 points) A certain $3 \times 3$ matrix $G$ has the following property: the matrix equation

$$G\vec{x} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$$

has exactly one solution. Based on this information, what can say about (if anything) about the number of solutions to the matrix equation $G\vec{x} = \begin{pmatrix} 0 \\ 1 \\ \pi \end{pmatrix}$.

Justify your answer completely.

(b) (2 points) Let $P$ be the set of points on the plane $x_1 + x_2 + x_3 = 0$ inside $\mathbb{R}^3$. Give an example of a linearly independent set of vectors in $P$ that is not a spanning set for $P$.

(c) (2 points) Let $P$ be the set of points on the plane $x_1 + x_2 + x_3 = 0$ inside $\mathbb{R}^3$. Give an example of a spanning set for $P$ that is not linearly independent.
8. For the following questions, no justifications are required.

(a) (2.5 points) Suppose you have a set of linearly dependent set of \( n \) vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) in \( \mathbb{R}^n \) with \( n \geq 2 \). Only one of the following four statements is true. Choose the right option.

i. Whenever there are scalars \( c_1, \ldots, c_n \) such that \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n = \vec{0} \), one of the scalars \( c_1, \ldots, c_n \) has to be non-zero.

ii. Whenever there are scalars \( c_1, \ldots, c_n \) such that \( c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_n \vec{v}_n = \vec{0} \), all of the scalars \( c_1, \ldots, c_n \) has to be non-zero.

iii. The set of \( n - 1 \) vectors \( \{ \vec{v}_2, \ldots, \vec{v}_n \} \) is linearly independent.

iv. There is a vector \( \vec{w} \) in \( \mathbb{R}^n \) that is not in the vector space spanned by \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \).

(b) (2.5 points) Only one of the following statements is false. Choose the statement that is false.

i. \( \left( \frac{\pi}{e} \right) \) is in the spanning set generated by the vectors \( \{(1, 0), (0, 1)\} \).

ii. \( \left( 0, 0, \sqrt{3} \right) \) is in the spanning set generated by the vectors \( \{(1, \sqrt{3}), (2, \sqrt{3})\} \).

iii. \( \left( 1, -1 \right) \) is in the spanning set generated by the vector \( \{(0, 0)\} \).

(c) (2.5 points) Suppose \( A \) is a \( 3 \times 3 \) matrix with the following property: The matrix equation \( A\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \) has a unique solution.

Consider the following statements:

Statement I: \( \det(A) \neq 0 \).

Statement II: The dimension of \( \text{Nullspace}(A) \) equals three.

Choose the correct option.

i. Only Statement I is true. Statement II is false.

ii. Only Statement II is true. Statement I is false.

iii. Both Statement I and Statement II are true.

iv. Both Statement I and Statement II are false.

(d) (2.5 points) Consider the following statements:

Statement A: Whenever \( A \) and \( B \) are \( n \times n \) skew-symmetric matrices, the matrix \( AB \) must be a symmetric matrix.

Statement B: Whenever \( A \) and \( B \) are \( n \times n \) skew-symmetric matrices, the matrix \( AB \) must be a skew-symmetric matrix.

Choose the correct option.

i. Only Statement A is true. Statement B is false.

ii. Only Statement B is true. Statement A is false.

iii. Both Statement A and Statement B are true.

iv. Both Statement A and Statement B are false.