Midterm #2 for Math 240, Spring 2017

Name (printed): ____________________________

My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this midterm examination.

Signature: ____________________________ Date: ____________________________

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- You have 80 minutes for this exam.
- Please show ALL your work on this exam paper. Partial credit will be awarded where appropriate. Answers with little or no justification will get no credit.
- You may use both sides of one 8.5 by 11 inch sheet of notes.
- NO books, laptops, cell phones, calculators, or any other electronic devices may be used during the exam.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.
- Box your final answers.
1. Consider the following $2 \times 2$ matrix.

$$A = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}.$$ 

Answer the following questions.

(a) (4 points) Compute the characteristic polynomial of $A$. Find the eigenvalues $\lambda_1$ and $\lambda_2$ of $A$.

(b) (6 points) Compute the set of eigenvectors for $\lambda_1$ and $\lambda_2$. 
(c) (5 points) Find an invertible matrix $S$ and a diagonal matrix $D$ such that $A = S^{-1}DS$.

(d) (5 points) Compute $e^{At}$. 
2. (10 points) Consider the set $V$ of $2 \times 2$ symmetric matrices (which is a subspace of $M_2(\mathbb{R})$). Consider the following two bases for $V$:

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right\}. $$

Find the change of basis $P_{B \to C}$ matrix.
3. Consider the linear transformation $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ defined below

$$T (p(x)) = \begin{pmatrix} p(0) \\ p(-1) \\ p(1) \end{pmatrix},$$

for all polynomials $p(x)$ in $P_2(\mathbb{R})$.

Here, $P_2(\mathbb{R})$ is the vector space consisting of polynomials in one variable (say $x$) with degree less than or equal to 2. Answer the following questions:

(a) (7 points) Find the matrix associated to the linear transformation $T$ with respect to the standard bases for $P_2(\mathbb{R})$ and $\mathbb{R}^3$.

(b) (3 points) Is the transformation $T$ invertible? Justify your answer completely.
4. (10 points) Consider a $3 \times 3$ non-diagonalizable matrix $D$ with the following characteristic polynomial:

$$p(\lambda) = (2 - \lambda)^2(1 - \lambda).$$

Answer the following questions:

(a) (5 points) Find $\text{Rank}(D - 5I_3)$.

(b) (5 points) Find $\text{Rank}(D - 2I_3)$. 
5. Consider a $3 \times 3$ matrix $G$ with the following properties:

$$\text{Rank}(G) = 2, \quad \text{Nullspace}(G + I_3) = \text{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\}, \quad \text{Nullspace}(G - I_3) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$ 

Answer the following questions:

(a) (5 points) Find the characteristic polynomial of $G$.

(b) Is the matrix $G$ diagonalizable? If yes, find a diagonal matrix similar to $G$. Justify your answer completely.
(c) (5 points) Determine whether the system

\[ G\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \]

is consistent. Justify your answer completely.
6. Are the following statements true or false?
(If the statement is true, provide complete justification as to why the statement is true. If the statement is false, provide an explicit counterexample to illustrate why the statement is false.)

(a) (5 points) Every invertible $2 \times 2$ matrix is diagonalizable.

(b) (5 points) If $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$ that satisfies the property $A^2 = A$, then $\lambda$ must equal either zero or one.
(c) (5 points) For all $n \times n$ matrices $A$ and $B$, we have

$$\text{Nullity}(AB) \geq \text{Nullity}(B).$$
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