1. This exam has three parts.

2. Part I has 2 questions about examples of matrices. (8 points each). For each of the question, either give an example or explain why none exists.

3. Part II has 3 (True/False) questions (8 points each). You need to give brief justification for your answer. If true, you can quote a relevant definition or theorem from the textbook or the lecture. If false, provide an example, illustration, or brief explanation of why the statement is false.

4. Part III has 5 standard problems. (12 points each).

5. Turn off and put away your cell phone.


7. Read each question carefully, and answer each question completely.

8. Show all of your work; no credit will be given for unsupported answers.

9. Write all your solutions clearly and legibly; no credit will be given for illegible solutions.

10. If any question is not clear, ask for clarification.

11. A vector \((a, b, c)\) means a column vector \[
\begin{bmatrix}
  a \\
  b \\
  c
\end{bmatrix}
\]
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Part I

For each of the following, give an example or explain why none exists.

**Problem I - 1.** A $2 \times 3$ matrix $A$ that has exactly one right inverse.

**Problem I - 2.** Two $n \times n$ matrices $A$ and $B$ such that $A$ and $AB$ are invertible matrices, but $B$ is not.
Part II

True or False. You need to give brief justification for your answer. If true, you can quote a relevant definition or theorem from the textbook or the lecture. If false, provide an example, illustration, or brief explanation of why the statement is false.

Problem II - 1. Five vectors in $P_3$ must be linearly independent.

Problem II - 2. If $A$ and $B$ are skew-symmetric $n \times n$ matrices, then $AB$ is symmetric.

Problem II - 3. The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x) = 0$ is a vector space.
Part III

Problem III - 1. Use Gaussian elimination to determine the solution set to the given system. Write down the elementary row operations explicitly. You won’t get full credits if you use other methods.

\[ \begin{align*}
    x_1 + 2x_2 - x_3 + x_4 &= 1 \\
    2x_1 + 4x_2 - 2x_3 + 2x_4 &= 2 \\
    5x_1 + 10x_2 - 5x_3 + 5x_4 &= 5
\end{align*} \]
Problem III - 2. Let

\[
A = \begin{bmatrix}
1 & i & 2 \\
1 + i & -1 & 2i \\
2 & 2i & 5
\end{bmatrix}
\]

It is known that $A$ is invertible. Use Gaussian elimination to find the first and the second column of $A^{-1}$. Write down the elementary row operations explicitly. You won’t get full credits if you use other methods.
Problem III - 3. The Vandermonde determinant of 3 variables is defined by

\[ V(r_1, r_2, r_3) = \begin{vmatrix} 1 & r_1 & r_1^2 \\ 1 & r_2 & r_2^2 \\ 1 & r_3 & r_3^2 \end{vmatrix} \]

Show that \( V(r_1, r_2, r_3) = (r_2 - r_1)(r_3 - r_1)(r_3 - r_2) \). (Hint: You may use row operations to simplify the determinant, and then use the property P9: \( \det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \det D \).)
Problem III - 4. If $p_1(x) = x - 4$ and $p_2(x) = x^2 - x + 3$, determine whether $p(x) = 2x^2 - x + 2$ lies in span\{p_1, p_2\}.
Problem III - 5. a) Use Problem III-3 to compute the Wronskian of the functions $f_1, f_2, f_3$.

$$f_1(x) = e^{r_1x}, f_2(x) = e^{r_2x}, f_3(x) = e^{r_3x}$$

b) Under which conditions on $r_1, r_2, r_3$, is the subset $\{f_1, f_2, f_3\}$ linearly dependent?
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