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3.2 Properties of det

4.1–4.2 Vector spaces
We usually use properties of $\det$ to compute

1) $T$ triangular

$$T = \begin{bmatrix}
a_{11} & a_{22} & * \\
& & \\
0 & & a_{nn}
\end{bmatrix}$$

or

$$\begin{bmatrix}
a_{11} & a_{22} & 0 \\
& & \\
* & & a_{nn}
\end{bmatrix}$$
\[
\det T = \prod_{i=1}^{n} a_{ii} = a_{11} a_{22} a_{33} \ldots a_{nn}
\]

2) We know how \(\det\) changes under ERO's.

\[\begin{align*}
&\text{P1} & A & \xrightarrow{P_{ij}} & B \\
&\det B &= -\det A
\end{align*}\]

\[\begin{align*}
&\text{P2} & A & \xrightarrow{M_{i}(k)} & B \\
&\det B &= k \det A
\end{align*}\]

\[\begin{align*}
&\text{P3} & A & \xrightarrow{A_{ij}(k)} & B \\
&\det A &= \det B
\end{align*}\]
\( p2 \rightarrow p4: \)
\[
\det(kA) = k^n \det(A)
\]

\( k \) A means every row is multiplied by \( k \)
Example

\[
\begin{vmatrix}
-1 & -4 & 2 \\
3 & 6 & 1 \\
4 & 2 & -1 \\
\end{vmatrix}
\]

\[
P_3 = \begin{vmatrix}
-1 & 4 & 2 \\
0 & -6 & 7 \\
0 & -14 & 7 \\
\end{vmatrix}
\]

\[
P_2 = \begin{vmatrix}
7 & -1 & 4 & 2 \\
0 & -6 & 7 \\
0 & -2 & 1 \\
\end{vmatrix}
\]
\[
P \begin{bmatrix} -1 & -4 & 2 \\ 0 & -2 & 1 \\ 0 & -6 & 7 \end{bmatrix}
\]

\[
P^3 \begin{bmatrix} -1 & -4 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 4 \end{bmatrix}
\]

\[
= -7 \times (-1) \times (-2) \times 4
\]

\[
= -56
\]
$p_1, p_2, p_3 \rightarrow \text{Theorem}$

$A \xrightarrow{\text{ERO}} R \xrightarrow{\text{REFORM}}$

$A$ invertible $\iff R$ invertible

$\det A \neq 0 \iff \det R = 1$

$P_1, P_2, P_3$. 

$\bigcup R$ triangular
P5 \quad \det A^T = \det A

P6. \quad \text{Look at } A \text{ row by row}

A = \begin{bmatrix}
1
\vdots
\beta_i
\alpha_i
\end{bmatrix}

\text{If} \quad \overline{a}_i = \overline{\beta}_i + \overline{r}_i

B = \begin{bmatrix}
\overline{1}_1
\overline{1}_2
\vdots
\overline{1}_n
\end{bmatrix}

C = \begin{bmatrix}
\overline{1}_1
\overline{1}_2
\vdots
\overline{1}_n
\end{bmatrix}
then \( \det(A) = \det B + \det C \)

P7 If \( A \) has a row of zeros, \( \det A = 0 \)

P8 If two rows of \( A \) are scalar multiples of each other, then \( \det A = 0 \)

P9 \( \det(AB) = \det A \det B \)

P10 If \( A \) is invertible, then

\[ \det(A^{-1}) = \frac{1}{\det A} \]
Because of P5

P6 P7 P8 are correct
if we change rows → columns.

\[
\begin{vmatrix}
A & B \\ 0 & C
\end{vmatrix} = |A| |C|
\]

matrix in blocks.
Example

Find all $x$ such that

$$\begin{vmatrix} x^2 & x & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 0$$

Row 2 $= 1^2 \ 1 \ 1$

$x = 1 \rightarrow$ two rows equal by PS

$\det = 0$.

Row 3 $= 2^2 \ 2 \ 1$

$x = 2 \Rightarrow \det = 0$. 
det is a quadratic function of $x$
at most two roots

$x = 1$ or $2$. 
Linear Algebra

Key: We care about the behavior/properties, not the objects!

OOP

Interface

different classes
Basic object

vectors / vector spaces

$\mathbb{R}^n$ space of matrices

vector space

space of functions
The common behavior

addition

scalar multiplication

that satisfy axioms

What we care about
Example

\[ \mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_i \in \mathbb{R}, i = 1, 2, 3 \right\} \]

addition

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}
\]

scalar multiplication \( k \in \mathbb{R} \)

\[
k \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \end{bmatrix}
\]
Example

Vectors in space

Addition

Scalar multiplication

\[ \vec{v}, 3\vec{v} \]
Consider the solution of

\[ X_1 - X_2 + 2X_3 = 0 \]
\[ 2X_1 - 2X_2 + 4X_3 = 0 \]
\[ 3X_1 - 3X_2 + 6X_3 = 0 \]

\[ A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ 3 & -3 & 6 & 0 \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ X_1 - X_2 + 2X_3 = 0 \]
\[ X_2, X_3 \text{ free} \quad X_1 = X_2 - 2X_3 \]
$\mathbf{v}_1 = (1, 1, 0)$ a solution

$\mathbf{v}_2 = (-2, 0, 1)$ also a solution

addition

$\mathbf{v}_1 + \mathbf{v}_2 = (-1, 1, 1)$ also a solution

scalar multiplication

$k \mathbf{v}_1 = (k, k, 0)$ also a solution
Define the solution set

\[ N(A) = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \} \]
Example

$2 \times 2$ matrices.

Addition

\[
\begin{bmatrix}
    a_1 & b_1 \\
    c_1 & d_1
\end{bmatrix}
+ \begin{bmatrix}
    a_2 & b_2 \\
    c_2 & d_2
\end{bmatrix}
= \begin{bmatrix}
    a_1 + a_2 & b_1 + b_2 \\
    c_1 + c_2 & d_1 + d_2
\end{bmatrix}
\]

Scalar multiplication

\[
k \begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} = \begin{bmatrix}
    ka & kb \\
    kc & kd
\end{bmatrix}
\]
Example

functions of $x$

addition

$$f_1 = e^x + x^2$$

$$f_2 = \sin x$$

$$f_1 + f_2 = e^x + x^2 + \sin x$$

scalar multiplication

$kf$
What do the above examples have in common?

If you want to define something that include all the examples, how would you define it?
Vector space
A nonempty set $V$
(whose elements are called vectors)

Two operations
1) addition
2) scalar multiplication
with scalars in $F (\mathbb{R} / \mathbb{C})$
such that $A1 = A10$

(Definition 4.2.1)
We care about the behaviors and properties (Two operation $A1 - A10$)
We don't care about what elements look like
(Different vector spaces look very differently)

To Do At Home
Read Definition 4.2.1
at least once
More examples

1. \( \mathbb{R}^n \): scalars in \( \mathbb{R} \)
   real vector space

2. \( \mathbb{C}^n = \{ (z_1, \ldots, z_n) \mid z_i \in \mathbb{C} \} \)
   scalars in \( \mathbb{C} \)
   complex vector space
Can you multiply vectors in $\mathbb{C}^3$ with scalars in $\mathbb{R}$?

Yes, but then it's a different vector space from $\mathbb{C}^3$.

Scalar multiplication different $\Rightarrow$ different vector space even the underlying set $V$ the same!
3. All $m \times n$ matrices
   $A + B$
   $kA$
   $M_{m \times n}(\mathbb{R})$ or $M_{m \times n}(\mathbb{C})$

4. All real/complex functions over some interval $I$
   $f + g$
   $kf$ for $k \in \mathbb{R}$ or $\mathbb{C}$. 
(5) $N(A) \subset \mathbb{R}^n$

A mxn matrix
is a subspace of $\mathbb{R}^n$

Definition (subspace)

$S$ a nonempty subset of
a vector space $V$ If $S$ itself
a vector space under the
addition and scalar multiplication
as used in $V$ then $S$ is
called a subspace of $V$
How to check a subspace?

No need to check A3 - A10 since V is already a vector space.

Theorem (subspace)
S a nonempty subset of V. Then S is a subspace of V.

⇒ S is closed under addition and scalar multiplication
Most of the examples in our course are subsets of Example (1) – (4)

Therefore to check whether a vector space

= check closed under addition

and scalar multiplication
Example

\[(1, -1, 2)\]

A plane in \(\mathbb{R}^3\) \(\perp (1, -1, 2)\)

Take two vectors \(\vec{u}_1, \vec{u}_2\) in the plane \(\vec{u}_1 + \vec{u}_2\) still in the plane
Take any vector \( \mathbf{v} \) in the plane and any number \( k \).

\( k\mathbf{v} \) is in the plane.

Not Example

plane \( \{ (x, y, z) \mid x - y + 2z = 1 \} \)

(1, 0, 0) is in the plane.

0 \cdot (1, 0, 0) = (0, 0, 0) not in the plane.

Not a vector space.
\[(1, 0, 0) \text{ in the plane}\]
\[(0, -1, 0) \text{ in the plane}\]
\[(1, 0, 0) + (0, -1, 0)\]
\[= (1, -1, 0) \text{ not in the plane}\]

Proposition

If \( v \) is not in \( V \),
then \( V \) is not a vector space.
Example:

\[ x_1 + 2x_2 - x_3 = 0 \]
\[ 2x_1 + 5x_2 - 4x_3 = 0 \]

Verify the solution set is a vector space.

\[ A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -4 \end{bmatrix} \]

Let the solution set \( S \) be \( N(A) \)
\[ A^\# = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -4 & 0 \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \]

\[ x_1 + 2x_2 - x_3 = 0 \]
\[ x_2 - 2x_3 = 0 \]
\[ x_3 \text{ free } \]
\[ x_3 = r \]
\[ x_1 = -3r \]
\[ x_2 = 2r \]

\[ N(A) = \{ (-3r, 2r, r) \mid r \in \mathbb{R} \} \]
1) $N(A)$ nonempty \\
(0, 0, 0) in N(A)

2) closed under addition \\
$\overrightarrow{x} = (-3r, 2r, r)$ \\
$\overrightarrow{y} = (-3s, 2s, s)$ \\
$\overrightarrow{x} + \overrightarrow{y} = (-3(r+s), 2(r+s), r+s)$ \\
still in $N(A)$

3) closed under scalar multiplication \\
$\overrightarrow{x} = (-3r, 2r, r)$ $k \in \mathbb{R}$ \\
$k\overrightarrow{x} = (-3kr, 2kr, kr)$ in N(A)
By theorem (subspace) \( N(A) \) is a subspace of \( \mathbb{R}^3 \) thus a vector space.

A \( m \times n \) matrix with entries in \( F \) (\( F = \mathbb{R} \) or \( \mathbb{C} \))

Define

\[ N(A) = \{ \mathbf{x} \in F^n \mid A\mathbf{x} = \mathbf{0} \} \]

is called the null space of \( A \).
Theorem

The null space $N(A)$ is a subspace of $F^n$. 