2.5 Gaussian Elimination
2.6 Inverse
Solve $A\bar{x} = \bar{b}$

**Gaussian Elimination**

1) $A\# = \begin{bmatrix} A & \bar{b} \end{bmatrix}$

2) $A\# \rightarrow R\# \text{ RE form}$

3) back substitution

**Gauss–Jordan Elimination**

1) $A\# = \begin{bmatrix} A & \bar{b} \end{bmatrix}$

2) $A\# \rightarrow R\# \text{ RRE form}$

3) back substitution
Example: Gaussian

\[3x_1 - 2x_2 + 2x_3 = 9\]
\[x_1 - 2x_2 + x_3 = 5\]
\[2x_1 - x_2 - 2x_3 = -1\]

1) \[A \neq \begin{pmatrix} 3 & -2 & 2 & 9 \\ 1 & -2 & 1 & 5 \\ 2 & -1 & -2 & -1 \end{pmatrix}\]

2) \[P^{-1} \begin{pmatrix} 1 & -2 & 1 & 5 \\ 3 & -2 & 2 & 9 \\ 2 & -1 & -2 & -1 \end{pmatrix}\]

\[A_{12}(-3) \begin{pmatrix} 1 & -2 & 1 & 5 \\ 0 & 4 & -1 & -6 \\ 0 & 3 & -4 & -11 \end{pmatrix}\]

\[A_{13}(-2) \begin{pmatrix} 1 & -2 & 1 & 5 \\ 0 & 4 & -1 & -6 \\ 0 & 3 & -4 & -11 \end{pmatrix}\]
$A_{32} (-1)$

\[
\begin{bmatrix}
1 & -2 & 1 & 5 \\
0 & 1 & 3 & 5 \\
0 & 3 & -4 & -11
\end{bmatrix}
\]

$A_{23} (-3)$

\[
\begin{bmatrix}
1 & -2 & 1 & 5 \\
0 & 1 & 3 & 5 \\
0 & 0 & -13 & -26
\end{bmatrix}
\]

$M_3 (-\frac{1}{13})$

\[
\begin{bmatrix}
1 & -2 & 1 & 5 \\
0 & 1 & 3 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

3) \(x_1 - 2x_2 + x_3 = 5\)

\(x_2 + 3x_3 = 5\)

\(x_3 = 2\)
\begin{align*}
  x_3 &= 2 \\
  x_2 &= -1 \\
  x_1 &= 1
\end{align*}

Unique solution \((1, -1, 2)\)
Example (Gauss–Jordan)

\[ \begin{align*}
5x_1 - 6x_2 + x_3 &= 4 \\
2x_1 - 3x_2 + x_3 &= 1 \\
4x_1 - 3x_2 - x_3 &= 5
\end{align*} \]

1) \[ A^\# = \begin{bmatrix}
5 & -6 & 1 & 4 \\
2 & -3 & 1 & 1 \\
4 & -3 & -1 & 5
\end{bmatrix} \]

2) \[ A_{31}(-1) \quad A_{12}(-2) \quad A_{13}(-4) \]
\[ \begin{bmatrix}
1 & -3 & 2 & -1 \\
2 & -3 & 1 & 1 \\
4 & -3 & -1 & 5
\end{bmatrix} \]
\[ \begin{bmatrix}
1 & -3 & 2 & -1 \\
0 & 3 & -3 & 3 \\
0 & 9 & -9 & 9
\end{bmatrix} \]
$M_2 \left(\frac{1}{3}\right) \begin{pmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 9 & -9 & 9 \end{pmatrix}$

$A_{23}(-9) \xrightarrow{\text{Gaussian}} \begin{pmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$A_{21}(3) \xrightarrow{\text{Gaussian}} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
\[ x_1 - x_3 = 2 \]
\[ x_2 - x_3 = 1 \]

2x3 system

\[ x_1 = 2 + x_3 \]
\[ x_2 = 1 + x_3 \]

\[ x_1, x_2 \text{ determined by } x_3 \]

\[ x_3 \text{ free} \]

\[ x_3 = t \]

\[ x_1 = 2 + t \]
\[ x_2 = 1 + t \]

\[ x_3 \text{ is called free variable} \]
The solution set is a line

\[(2+t, 1+t, t)\]

Essentially the intersection of two planes, although we started from a 3x3 system.

Compare with the first example.
X₃ becomes free because there is no leading 1 for X₃.
A column of $A$ (not $A^T$) without leading 1 is called a free column.

$\#\text{free column} = \#\text{free variable}$

\[ n - r \quad \text{dimension of the solution} \]
Example

\[ \begin{align*}
X_1 + X_2 - X_3 + X_4 &= 1 \\
2X_1 + 3X_2 + X_3 &= 4 \\
3X_1 + 5X_2 + 3X_3 - X_4 &= 3
\end{align*} \]

\[
\begin{bmatrix}
1 & 1 & -1 & 1 & 1 \\
2 & 3 & 1 & 0 & 4 \\
3 & 5 & 3 & -1 & 3
\end{bmatrix}
\]

\[A_{12}(-2)\]

\[A_{13}(-3)\]

\[
\begin{bmatrix}
1 & 1 & -1 & 1 & 1 \\
0 & 1 & 3 & -2 & 2 \\
0 & 2 & 6 & -4 & 0
\end{bmatrix}
\]
\[ A_{23}(-2) \rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \]

The last row

0 \( x_1 \) + 0 \( x_2 \) + 0 \( x_3 \) + 0 \( x_4 \) = -4

0 = -4

Impossible

No solution. We call it inconsistent

\[ \text{rank } A = 2 \quad \text{rank } A^\# = 3 \]
The last column is a pivot column

\[ \Rightarrow \text{rank } (A^\#) = \text{rank } (A) + 1 \]

In this case we would have

\[
\begin{bmatrix}
  \ast & \ast \\
  0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]

\( \Rightarrow o = 1 \)

inconsistent.
inconsistent $\iff \text{rank}(A^\#) > \text{rank}(A)$

Theorem 2.5.9 (Significance of rank)

$m \times n$ system $A \bar{x} = \bar{b}$

\[
\begin{align*}
\text{rank}(A) &= r \\
\text{rank}(A^\#) &= r^\#
\end{align*}
\]

1) $r < r^#$ inconsistent.

2) $r = r^#$ consistent

a) unique solution $\iff r = n$

b) infinite solutions $\iff r < n$
\[ r = \# \text{ independent equations} \]

\[ = \# \text{ independent relations among unknowns} \]

\[ n - r = \# \text{ freedoms} \]

EROS preserve rank

( Solution sets are the same)

Row equivalent matrices have the same rank
Definition

\[ A \vec{x} = \vec{b} \]

is homogeneous if \( \vec{b} = \vec{0} \)

is nonhomogeneous if \( \vec{b} \neq \vec{0} \)

A homogeneous system is always consistent.

For example, \( \vec{x} = \vec{0} \) is a solution.

\[ \text{rank } A = \text{rank } [ A \ \vec{0} ] \]
Very important

If $A$ is an $m \times n$ matrix
\[ r = \text{rank}(A) \]
\[ r \leq m \]
\[ r \leq n \]

Consider the RE form $R$
also an $m \times n$
Each row has at most one leading 1
\[ r \leq m \]
Each column has at most one leading 1
\[ r \leq n \]
Corollary

A homogeneous $m \times n$ system with $m < n$ has an infinite number of solutions.

$r \leq m < n$

2) b) in Theorem 2.5.9

The cases when "=" hold are very special.
If \( r = m \) \( A \) is called full row rank.

If \( r = n \) \( A \) is called full column rank.

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**full row rank**

\[ r = m \leq n \]

What can we say about

\[ A \vec{x} = \vec{b} \]
\[ A^\# = \begin{pmatrix} A & b \end{pmatrix} m \times (n+1) \]

\[ r^\# = \text{rank } (A^\#) \leq m = r \]

and \( r \leq r^\# \)

So \( r^\# = r \) for any \( b \)!

Whatever \( b \) is

\[ A \tilde{x} = b \]

always has solutions (consistent)

Choose \( b = \begin{bmatrix} 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 1 \\ \vdots \end{bmatrix} \ldots \)

the standard basis \( \{e_i\} \) of \( \mathbb{R}^m \)
There exist solutions

\[ A \alpha_1 = \vec{e}_1 \]
\[ A \alpha_2 = \vec{e}_2 \]
\[ \vdots \]
\[ A \alpha_m = \vec{e}_m \]

Let \( B = [\alpha_1 \alpha_2 \ldots \alpha_m] \) \( n \times m \)

\( AB = [ A\alpha_1 \ A\alpha_2 \ldots \ A\alpha_m ] \)
\[ = [ \vec{e}_1 \ \vec{e}_2 \ldots \ \vec{e}_m ] \]
\[ = I_m \]
Proposition

If \( A \) has \( r = m \), there is an \( n \times m \) matrix \( B \) such that \( AB = I_m \).

On the other hand

If there is \( B \) \( n \times m \) such that \( AB = I_m \),

For any \( \vec{b} \in \mathbb{R}^m \) \( m \)-vector

\[
A(B \vec{b}) = (AB) \vec{b} = I_m \vec{b} = \vec{b}
\]
$B \bar{c}$ is a solution to

$$A \bar{x} = \bar{b}$$

Whatever $\bar{b}$ is

$\text{rank } [A \bar{b}] = \text{rank } A$

$\Rightarrow \text{rank } A = m$

If $AB = I_m$

we call $B$ a right inverse of $A$. 
Proposition

A full row rank \(\Rightarrow\) A has a right inverse

Similarly

\[ A_{m \times n} \text{ if there is } C_{n \times m} \text{ such that } \]

\[ CA = I_n \]

C is called a left inverse of A
Proposition

A full column rank $\iff$ $A$ has a left inverse.

Proof

$r = n \iff A\overline{x} = \overline{b}$ has at most one solution for any $\overline{b}$

$(\Rightarrow)$ If $A\overline{x} = A\overline{\beta}$

"$\iff$" then $\overline{x} = \overline{\beta}$.

If $CA = I_n$ and $A\overline{x} = A\overline{\beta}$

\[ \overline{x} = (CA)\overline{x} = C(A\overline{x}) = C(A\overline{\beta}) = (CA)\overline{\beta} = \overline{\beta} \]
"⇒"

1. If \( n = r \)

What's the RRE of \( A \)?

\[
\begin{bmatrix}
1 \\
\vdots \\
0
\end{bmatrix}
= \begin{bmatrix}
I_n \\
0
\end{bmatrix}
\]

Fact: ERO's can be written as left multiplication with some matrices

\[⇒ \quad C' A = \begin{bmatrix}
\frac{1}{n} \\
0
\end{bmatrix} \]

\( C = \) the first \( n \) rows of \( C' \)
Example

\[ A = \begin{bmatrix} -3 & 4 \\ 4 & 6 \end{bmatrix} \]

\[ B_1 = \frac{1}{9} \begin{bmatrix} -11 & -10 & 16 \\ -7 & 8 & -11 \end{bmatrix} \]

\[ B_2 = \frac{1}{2} \begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix} \]

Check

\[ B_1 A = I_2 \quad AB_1 ? \]
\[ B_2 A = I_2 \quad AB_2 ? \]

Left inverse is not unique. You may have infinitely many.
To Do At Home

Can you produce infinitely many left inverses for the $A$ above from $B_1$ and $B_2$?

Hint: \( \frac{1}{2} B_1 + \frac{1}{2} B_2 \) ?

\( \frac{1}{3} B_1 + \frac{2}{3} B_2 \) ?

\( t B_1 + (1-t) B_2 \) ?
<table>
<thead>
<tr>
<th>Has right inverse</th>
<th>No right inverse</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Has left inverse</strong></td>
<td><strong>No left inverse</strong></td>
</tr>
<tr>
<td>invertible (inverse unique)</td>
<td>infinitely many left inverse</td>
</tr>
<tr>
<td>$n = m = r$</td>
<td>$m &gt; n = r$</td>
</tr>
<tr>
<td>$n &gt; m = r$</td>
<td>$r &lt; n$</td>
</tr>
<tr>
<td>$r &lt; m$</td>
<td></td>
</tr>
</tbody>
</table>
Most important case

A \( n \times n \) a square

1) If \( A \) has a left inverse then \( A \) has a right inverse (vice versa)

2) In that case left inverse unique, right inverse unique and they are equal

3) In that case we call it the inverse of \( A \)

denote \( A^{-1} \)
Proposition

A n x n square

If \(AB = I\), \(CA = I\)

then \(B = C\)

Proof

\[B = IB = (CA)B\]

\[= C(AB) = CI = C\]

It follows that left/right inverse is unique.
Definition

A n×n square

If there is n×n \( A^{-1} \) such that

\[
A(A^{-1}) = (A^{-1}) A = I
\]

then \( A^{-1} \) is called the inverse of \( A \). \( A \) is called invertible

Invertible = nonsingular

Square but not invertible = singular
How to find $A^{-1}$?

Recall we found the right inverse $B$ for $AB = I$?

Solve

$$A \beta_1 = \mathbf{e}_1$$
$$A \beta_2 = \mathbf{e}_2$$
$$\vdots$$
$$A \beta_m = \mathbf{e}_m$$

$m$ equations

$$B = [\beta_1 \ \beta_2 \ \cdots \ \beta_m]$$
Fortunately
We can solve these
in equations

\[ A \bar{x} = \bar{e}_i \text{ all at once} \]

\[ A^\# = \begin{pmatrix} A & I \end{pmatrix} \]

\[ = \begin{pmatrix} A & \bar{e}_1 & \bar{e}_2 & \cdots & \bar{e}_n \end{pmatrix} \]

\[ \begin{pmatrix} \text{EROs} \end{pmatrix} \]

\[ \begin{bmatrix} R & \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix} \]

R the RRE for A
\[ n = r \implies R = I_n \]

\( \vec{\beta}_i \) solves the equation

\[ I \vec{x} = \vec{a}_i \]

\[ \implies \vec{\beta}_i = \vec{a}_i \]

Algorithm

\[ A \rightarrow \begin{bmatrix} A & I \end{bmatrix} \]

\[ \begin{bmatrix} I & A^{-1} \end{bmatrix} \]

\[ \text{EROs} \]
Properties of $A^{-1}$

1. When $A$, $B$ invertible
   \[
   (A^{-1})^{-1} = A
   \]

2. $AB$ invertible
   \[
   (AB)^{-1} = B^{-1}A^{-1}
   \]

3. $A^T$ invertible
   \[
   (A^T)^{-1} = (A^{-1})^T
   \]

⚠️ You must be sure $A$ is invertible before you write $A^{-1}$

For any arbitrary matrix $B$, $B^{-1}$ doesn't make sense!
Theorem

A n x n square

The following are equivalent

a) A is invertible

b) A has a left inverse

c) A has a right inverse

d) $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$

e) $A\vec{x} = \vec{0}$ has only one solution

f) $r = n$

g) A is row-equivalent to I