Oct 11

Example

\[ A = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 \\
\end{pmatrix} \]
Last time

Rows in $A$ (Note not in $R$)

$\uparrow$

edges in the diagram

Linearly independent $\rightarrow$ No loop

Row 2 = Row 1 + Row 3

Linearly dependent $\rightarrow$ loops.

Since $\dim \text{C}(A^T) = 3$.

three linearly independent vectors give a basis
Looking for a basis for $C(A^T)$

→ look for 3 edges without a loop

A graph without a loop is called a tree

→ look for trees with 3 edges in

< >
For the left null space $N(A^T)$

$$A^T = \begin{bmatrix}
-1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$

$$A^T \overline{y} = \overline{0}$$

solutions:

\[-y_1 - y_2 = 0\]
\[y_1 - y_3 - y_4 = 0\]
\[y_2 + y_3 - y_5 = 0\]
\[y_4 + y_5 = 0\]
This is the picture for currents. By Kirchhoff's Current Law we have the system of equations.

\[
A^T = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[y_3, y_5 \text{ free variables}
\]
\[y_3 = 1 \quad y_5 = 0 \]
\[y_3 = 0 \quad y_5 = -1 \]
$N(\mathbf{A}^T)$ spanned by

\[
\begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
-1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]

arrow reversed

\[
\begin{pmatrix}
-1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]

\begin{pmatrix}
-1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]

independent loops

independent solutions.
has two independent loops

\[
\dim N(A^T) = 2
\]

Also we see \( N(A^T) \) no relation to

\[
R = \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

except \( \dim N(A^T) = \dim N(R^T) \)
$A \sim R$ row equivalent

$C(A^T) = C(R^T)$

$N(A) = N(R)$

$C(A)$ preserves linear dependence

$N(A^T) \quad N(R^T)$ no relation except

$\dim N(A^T) = \dim (R^T)$
\[ C(A) \]
pivot columns 1, 2, 3.
So \( C(A) \) has a basis

\[
\begin{bmatrix}
-1 \\
-1 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
0 \\
-1
\end{bmatrix}
\]

columns have no meanings.
4.1 Orthogonality

Recall
\( \vec{v}, \vec{w} \) vectors in \( \mathbb{R}^n \)

\( \vec{v} \perp \vec{w} \) (\( \vec{v} \) is orthogonal to \( \vec{w} \))

if \( \vec{v} \cdot \vec{w} = 0 \) dot product

or \( \vec{v}^T \vec{w} = 0 \) matrix product

Also we can say

\( \vec{v} \perp \vec{w} \) if \( \theta = \frac{\pi}{2} \)

\( \vec{v} \perp \vec{w} \) if \( \left\| \vec{v} + \vec{w} \right\|^2 = \left\| \vec{v} \right\|^2 + \left\| \vec{w} \right\|^2 \)
Definition
Two subspaces \( V \) and \( W \) of \( \mathbb{R}^n \) are orthogonal if every vector \( \vec{v} \) in \( V \) is orthogonal to every vector \( \vec{w} \) in \( W \).
Denote it by \( V \perp W \).

\[ \vec{v} \perp \vec{w} \text{ for } \vec{v} \in V, \vec{w} \in W. \]

The walls are orthogonal.
Basic example \( A_{mxn} \) matrix

Take \( \vec{x} \) from \( N(A) \)

Take any row \( \vec{\beta}_i \) in \( A \)

\[
A \vec{x} = \begin{bmatrix}
\vec{\beta}_1^T \\
\vec{\beta}_2^T \\
\vdots \\
\vec{\beta}_m^T
\end{bmatrix}
\begin{bmatrix}
\vec{x}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\vec{\beta}_1^T \vec{x} \\
\vec{\beta}_2^T \vec{x} \\
\vdots \\
\vec{\beta}_m^T \vec{x}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\]
\textbf{\underline{\textit{\( \vec{x} \perp \text{every row} \, \vec{\beta}_i \, \text{of} \, \vec{A} \)}}}

How about a linear combination of \( \vec{\beta}_1, \ldots, \vec{\beta}_m \)?

Say \( \vec{v} = a_1 \vec{\beta}_1 + \ldots + a_m \vec{\beta}_m \)

\[ \vec{v}^T \vec{x} = (a_1 \vec{\beta}_1^T + \ldots + a_m \vec{\beta}_m^T) \vec{x} \]

\[ = a_1 \vec{\beta}_1^T \vec{x} + \ldots + a_m \vec{\beta}_m^T \vec{x} \]

\[ = (a_1, \ldots, a_m) \begin{bmatrix} \vec{\beta}_1^T \vec{x} \\ \vdots \\ \vec{\beta}_m^T \vec{x} \end{bmatrix} \]

\[ = (a_1, \ldots, a_m) \vec{A} \vec{x} = 0 \]

\( \vec{v} \perp \vec{x} \)
Therefore any \( \vec{v} \) in \( \text{C}(A^T) \) is \( \perp \) any \( \vec{x} \) in \( \text{N}(A) \)

\[ \text{C}(A^T) \perp \text{N}(A) \]

Now consider \( A^T \)

\[ (A^T)^T = A \]

So \( \text{C}(A) \) the column space
for \( A \) is the row space
for \( A^T \)

\( \text{N}(A^T) \) the left null space
for \( A \) is the null space
for \( A^T \)
\[ \text{C}(A) \perp \text{N}(A^T) \]

We have even more

Definition The orthogonal complement of a subspace \( V \) contains every vector that is orthogonal to \( V \). We denote it by \( V^\perp \)

\[ V^\perp = \{ \overline{v} \in \mathbb{R}^n \mid \overline{v} \perp V \} \]

Proposition: \( V^\perp \) is a subspace

check (1) \( \perp \). (2)
Example

$N(A)$ is the orthogonal complement of $C(A^T)$

$A\vec{x} = \vec{0}$ means exactly

$\vec{x} \perp$ every row $\vec{b}_i$

That's the same as

$\vec{x} \perp C(A^T)$
Since \( V^\perp \) is a subspace, we can talk about \((V^\perp)^\perp\).

Proposition

If \( V \) is a subspace of a finite dimensional vector space \( \mathbb{R}^n \), then

\[(V^\perp)^\perp = V\]

We can't show this now. Instead, we show a special case.
\[ C(A^T) = N(A)^+ \]

Every vector \( \mathbf{u} \in C(A^T) \) is a linear combination of rows \( \mathbf{b}_1, \ldots, \mathbf{b}_m \) of \( A \).

Each \( \mathbf{b}_i \perp N(A) \)

So \( \mathbf{u} \perp N(A) \)

On the other hand, let \( \mathbf{v} \) be a vector in \( N(A)^+ \)

So \( \mathbf{v} \perp N(A) \)
Consider

\[ B = \begin{bmatrix} A \\ \vec{v} \end{bmatrix}_{m+1 \times n} = \begin{bmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_m \\ \vec{v} \end{bmatrix} \]

Since \( \vec{v} \perp N(A) \)

\[ N(B) = N(A) \]

\[ \dim N(B) = \dim N(A) = n - r \]

So \( \text{rank}(B) = r \)

That means \( \dim C(B^T) = r \)

\[ C(B^T) = C(A^T) \]

\( \vec{v} \) is in \( C(A^T) \)
\[ C(A^T) = N(A)^\perp \]

For the same reason
\[ C(A) = N(A^T)^\perp \]
\[ N(A^T) = C(A)^\perp \]

Lemma.
\[ V \cap V^\perp = \{ \overline{0} \} \]

Proof. Let \( \overline{z} \in V \cap V^\perp \)
so \( \overline{z} \perp \overline{z} \)
\[ ||\overline{z}||^2 = \overline{z} \cdot \overline{z} = 0 \]
So \( \overline{z} = \overline{0} \)
Now for any subspace \( V \) of \( \mathbb{R}^n \)

Choose a basis

\[ \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r \} \] of \( V \)

Regard \( \vec{u}_i \) as row vectors

\[ A = \begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_r \end{bmatrix} \] \( r \times n \) matrix

Then \( V = \text{C}(A^T) \)

So \( V^\perp = \text{C}(A^T)^\perp = N(A) \)

We know if \( \dim V = r \)

then \( \dim V^\perp = n - r \)
Proposition

If \( \{ \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_r \} \) is a basis for \( V \)
\[ \{ \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_{n-r} \} \] is a basis for \( V^\perp \)
then \( \{ \overline{u}_1, \ldots, \overline{u}_r, \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_{n-r} \} \) is a basis for \( \mathbb{R}^n \).

Proof. We have \( n \) vectors
\[ \{ \overline{u}_1, \overline{u}_2, \ldots, \overline{u}_r, \overline{v}_1, \ldots, \overline{v}_{n-r} \} \]
\( n = \dim \mathbb{R}^n \).
So it suffices to show that they are linearly independent.
Suppose we have a linear relation
\[ a_1 \vec{u}_1 + \ldots + a_r \vec{u}_r + b_1 \vec{v}_1 + \ldots + b_{n-r} \vec{v}_{n-r} = \vec{0} \]
So we have
\[ a_1 \vec{u}_1 + \ldots + a_r \vec{u}_r = -b_1 \vec{v}_1 - \ldots - b_{n-r} \vec{v}_{n-r} \]
The left hand side is in \( V \)
The right hand side is in \( V^\perp \)
So call it \( \ell \)
\( \ell \) is in \( V \cap V^\perp = \{ \vec{0} \} \)
Therefore
\[ a_1 \vec{u}_1 + a_2 \vec{u}_2 + \ldots + a_r \vec{u}_r = \vec{0} \]
\[ b_1 \vec{v}_1 + b_2 \vec{v}_2 + \ldots + b_{n-r} \vec{v}_{n-r} = \vec{0} \]
However \( \{ \overline{u}_1, \ldots, \overline{u}_r \} \)

is linearly independent

\[ a_1 = a_2 = \cdots = a_r = 0 \]

\( \{ \overline{v}_1, \ldots, \overline{v}_{n-r} \} \)

is linearly independent

\[ b_1 = b_2 = \cdots = b_{n-r} = 0 \]

It follows that

\( \{ \overline{w}_1, \ldots, \overline{w}_r, \overline{v}_1, \ldots, \overline{v}_{n-r} \} \)e linearly independent \( \square \).
Corollary

Each vector \( \overrightarrow{u} \in \mathbb{R}^n \) can be written as a sum

\[
\overrightarrow{u} = \overrightarrow{v} + \overrightarrow{v}^t
\]

such that \( \overrightarrow{v} \in V, \overrightarrow{v}^t \in V \)

Moreover, this decomposition is unique.

Proof: Just take the basis

\[
\{ \overrightarrow{w}_1, \ldots, \overrightarrow{w}_n \} \text{ of } V
\]

\[
\{ \overrightarrow{v}, \ldots, \overrightarrow{v}_{n-r} \} \text{ of } V^t
\]

\[
\{ \overrightarrow{u}_1, \ldots, \overrightarrow{u}_r, \overrightarrow{v}_1, \ldots, \overrightarrow{v}_{n-r} \}
\]
is a basis for $\mathbb{R}^n$. So $\mathbf{u} \in \mathbb{R}^n$ can be written as a linear combination

$$\mathbf{u} = a_1 \mathbf{u}_1 + \cdots + a_r \mathbf{u}_r + b_1 \mathbf{v}_1 + \cdots + b_{n-r} \mathbf{v}_{n-r}$$

Let $-\mathbf{v} = a_1 \mathbf{u}_1 + \cdots + a_r \mathbf{u}_r$

$$-\mathbf{v} = b_1 \mathbf{v}_1 + \cdots + b_{n-r} \mathbf{v}_{n-r}$$

So $\mathbf{u} = \mathbf{v} + (-\mathbf{v})$

and $\mathbf{v} \in V$, $(-\mathbf{v}) \in V^\perp$
Now if

\[ \overline{u} = \overline{v} + \overline{v}^\perp = \overline{w} + \overline{w}^\perp \]

\[ \overline{v} \in V, \overline{v}^\perp \in V^\perp \]

\[ \overline{w} \in V, \overline{w}^\perp \in V^\perp \]

(Two ways).

We have

\[ \overline{v} - \overline{w} = \overline{w}^\perp - \overline{v}^\perp \]

Again the left hand side is

in \( V \) the right hand side

is in \( V^\perp \)

So \( \overline{v} - \overline{w} = \overline{w}^\perp - \overline{v}^\perp \) is in \( V \cap V^\perp \)

\[ \overline{v} = \overline{w}, \overline{w}^\perp = \overline{v}^\perp. \]
Each vector $\overline{u} \in \mathbb{R}^n$

$\overline{u} = \overline{u}_r + \overline{u}_n$

for $\overline{u}_r \in C(A^T)$ $\overline{u}_n \in N(A)$

in a unique way.
Each vector $\vec{b}$ in $\mathbb{R}^m$

$$\vec{b} = \vec{b}_c + \vec{b}_n$$

for $\vec{b}_c \in \text{C}(A)$ and $\vec{b}_n \in \text{N}(A^T)$

in a unique way.