Oct 4

We have a finite set of vectors $\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n$ in the vector space $V$.

Definition

A linear relation among vectors $\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_n$ is any linear combination that evaluates to zero. That is the equality

$$a_1\overrightarrow{v}_1 + a_2\overrightarrow{v}_2 + \cdots + a_n\overrightarrow{v}_n = \overrightarrow{0}$$

for some scalars $a_1, a_2, \ldots, a_n$. 
Definition
A set of vectors $S$ is called \underline{linearly independent} if any linear relation among vectors from this set $S \quad a_1 \overrightarrow{v_1} + a_2 \overrightarrow{v_2} + \ldots + a_n \overrightarrow{v_n} = \overrightarrow{0}$ is trivial: $a_1 = a_2 = \ldots = a_n = 0$

Remark, I am not assuming $S$ to be a finite set. But we will see later that $S$ has to be finite if $V$ is a finite dimensional vector space.
Definition

A set of vectors $S$ is linearly dependent if there is some nontrivial linear relation. That is, there are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in S$ and $a_1, a_2, \ldots, a_n$ not all zeros such that

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \ldots + a_n \vec{v}_n = \vec{0}$$

Example

$S = \{ \vec{0} \}$

Linearly independent or dependent?
Example

\[ A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \]

The columns of \( A \) are linearly dependent since

\[ A \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \text{ nontrivial solution} \]
In fact

\[ N(A) \text{ is generated by} \]

\[
\begin{bmatrix}
-3 \\
1 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 3 \\
2 & 1 & 5 \\
1 & 0 & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 3 \\
2 & 1 & 5 \\
0 & 0 & 0
\end{bmatrix}
\]

Row 3 = Row 1

\[
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

\[ x_3 = 1 \rightarrow \begin{bmatrix}
-3 \\
1 \\
1
\end{bmatrix}, \quad r = 2 = 3 - 1 \]

\text{free}
However
\[ B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \]

The columns of \( B \) are linearly independent.

\[
\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \sim \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}
\]

\( N(B) = \{ \vec{0} \} \)
\( r = n = 0 \)

The only linear relation is trivial.
Full column rank

\[ R = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ N(A) = \{ \vec{0} \} \]

The columns are linearly independent.

Question: Can you have more than \( n \) vectors that are linearly independent in \( \mathbb{R}^n \)?

Think about how many pivots can we have at most in the matrix.
Span a vector space

Recall a vector space $W$ is spanned by $\overline{v}_1, \ldots, \overline{v}_n$ if $W$ consists of all linear combinations of $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$

We also say the set of vectors $\{\overline{v}_1, \ldots, \overline{v}_n\}$ span $W$. 
$C(A)$ is spanned by all columns of $A$

\[
\begin{pmatrix}
\frac{1}{2} \\
0 \\
\frac{2}{3}
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{3}{5} \\
\frac{1}{5}
\end{pmatrix}
\text{span} \quad C(A)
\]

\[
\begin{pmatrix}
0 \\
\frac{1}{6} \\
\frac{2}{6}
\end{pmatrix}
\text{also span} \quad C(B) \quad \text{if}
\]

$C(A)$

Adding more linear combinations does not change the span.

Definition: The row space of $A$ is the subspace of $\mathbb{R}^n$ spanned by the rows.
In other words the row space of $A$ is $C(A^T)$.

Question: To span $\mathbb{R}^n$ you need at least $n$ vectors.
Definition

$V$ a vector space over $F$.

A basis for $V$ is an ordered set, that is

1) linearly independent
2) also spans $V$.

A vector space is finite dim' if it has a finite basis.

Let $\beta = \{\overline{v}_1, \overline{v}_2, \ldots \overline{v}_n\}$
be a basis for $V$. 
\( B \) spans \( V \) means any vector \( \mathbf{b} \in V \) is a linear combination of 
\[ \overrightarrow{u}_1, \overrightarrow{u}_2, \ldots, \overrightarrow{u}_n. \]

\[ \mathbf{b} = a_1 \overrightarrow{u}_1 + a_2 \overrightarrow{u}_2 + \ldots + a_n \overrightarrow{u}_n. \tag{1} \]

\( B \) is linearly independent means this is the only way to write \( \mathbf{b} \) as a linear combination of \( \overrightarrow{u}_1, \ldots, \overrightarrow{u}_n \)

If \( \mathbf{b} = b_1 \overrightarrow{u}_1 + b_2 \overrightarrow{u}_2 + \ldots + b_n \overrightarrow{u}_n \), \( \tag{2} \)

\[ (1) - (2) \]
\[ \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{a}} = a_1 \overrightarrow{\mathbf{v}_1} + \cdots + a_n \overrightarrow{\mathbf{v}_n} - (b_1 \overrightarrow{\mathbf{v}_1} + \cdots + b_n \overrightarrow{\mathbf{v}_n}) \\
= (a_1 - b_1) \overrightarrow{\mathbf{v}_1} + (a_2 - b_2) \overrightarrow{\mathbf{v}_2} + \cdots + (a_n - b_n) \overrightarrow{\mathbf{v}_n} \]

This is a linear relation for \( \overrightarrow{\mathbf{v}_1}, \overrightarrow{\mathbf{v}_2}, \ldots, \overrightarrow{\mathbf{v}_n} \)

Since \( \overrightarrow{\mathbf{v}_1}, \overrightarrow{\mathbf{v}_2}, \ldots, \overrightarrow{\mathbf{v}_n} \) are linearly independent, this implies \( b_1 = a_1, \ b_2 = a_2, \ldots, b_n = a_n \)
\( B = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) is a basis means

There is one and only one way to write any vector \( \vec{v} \) as a linear combination of \( \{ \vec{v}_1, \ldots, \vec{v}_n \} \)

Example

\[
\begin{bmatrix}
1 \\ 0 \\ 0
\end{bmatrix},
\begin{bmatrix}
0 \\ 1 \\ 0
\end{bmatrix},
\begin{bmatrix}
0 \\ 0 \\ 1
\end{bmatrix}
\]

is a basis for \( \mathbb{R}^3 \).

Call it the standard basis for \( \mathbb{R}^n \).
Example A n x n invertible matrix. The columns of A → a basis for \( \mathbb{R}^n \)

1) \( N(A) = \{ 0 \} \)

The columns are linearly independent.

2) Any \( \mathbf{b} \in \mathbb{R}^n \) is a linear combination of the columns.

The columns span \( \mathbb{R}^n \).

⇒ The columns give a basis.
\[ A \overline{x} = \overline{b} \]

For any \( B \) there is one and only one solution \( \overline{x} = A^{-1} \overline{b} \)

First we know a basis \( B \) for \( \mathbb{R}^n \) must have \( n \) elements.

\[ \mathcal{B} = \{ \overrightarrow{v}_1, \ldots, \overrightarrow{v}_n \} \text{ is a basis for } \mathbb{R}^n \text{ exactly when the matrix } A = [ \overrightarrow{v}_1 \overrightarrow{v}_2 \overrightarrow{v}_n ] \text{ is invertible} \]
The pivot columns of $A$ are a basis for $\mathcal{C}(A)$.
The pivot rows of $A$ are a basis for $\mathcal{C}(A^T)$, the row space.

1) $B := \{v_1, \ldots, v_r\}$ the pivot columns
   $N(B) = \{\vec{0}\}$ linearly independent

2) Any free column $\leftrightarrow y_i$
   $y_i = 1$ the rest 0.
   the special solution
   $(a_1, a_2, \ldots, a_r, 0, \ldots, 0, 1, 0, \ldots, 0)$
   $a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_r\vec{v}_r + \vec{v}_i = \vec{0}$
\[ \overline{v_i} = -a_1 \overline{v_1} - a_2 \overline{v_2} \ldots - a_r \overline{v_r} \]

\[ \overline{v_i} \in C(B) \]

Example: Find the basis for the column and row space of

\[ R = \begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix} \]

\[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \] a basis for \( C(R) \)

\[ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \] a basis for \( C(R^T) \)
Question
Given 5 vectors in \( \mathbb{R}^7 \).
How to find a basis for the subspace they span?

I. Make them rows of a matrix \( A \) (\( 5 \times 7 \))
\( \rightarrow \mathbb{R} \) Find the nonzero rows of \( R \) \( C(A^T) = C(R^T) \)

II. Make them columns of a matrix \( B \) (\( 7 \times 5 \))
Find the pivot columns of \( B \) (not of \( R \))
Dimension of a vector space

Any basis for \( \mathbb{R}^n \) has \( n \) elements, we say the dim' of \( \mathbb{R}^n \) is \( n \).

\[ n=1 \quad \text{line} \]
\[ n=2 \quad \text{plane} \]
\[ n=3 \quad \text{3D space} \]

Theorem

If \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \) and \( \vec{w}_1, \ldots, \vec{w}_n \) are both bases for the same vector space, then \( m = n \).
Proof: Suppose \( n > m \)
we want to reach a contradiction.
\( \overline{v}_1 \ldots \overline{v}_m \) basis.

\[ \Rightarrow \text{Any } \overline{w}_j \text{ is a linear combination of } \overline{v}_1 \ldots \overline{v}_m \text{ in a unique way} \]

\[ \overline{w}_j = a_{ij} \overline{v}_1 + a_{i2} \overline{v}_2 + \ldots + a_{im} \overline{v}_m \]

\[
\begin{bmatrix}
\overline{w}_1 \\
\vdots \\
\overline{w}_n
\end{bmatrix} = 
\begin{bmatrix}
\overline{v}_1 & \overline{v}_2 & \ldots & \overline{v}_m
\end{bmatrix}
\begin{bmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \ldots & a_{mn}
\end{bmatrix}
\]

\[ = VA \]

\[ A = \begin{bmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \ldots & a_{mn}
\end{bmatrix} \quad m \times n \quad n > m \]
So in particular $n > r$

$A\vec{x} = \vec{0}$

has nontrivial solution $\vec{x}$

$W\vec{x} = VA\vec{x} = V(A\vec{x}) = \vec{0}$

for $\vec{x} \neq \vec{0}$

$N(W) \neq \{ \vec{0} \}$

$\Rightarrow \vec{w}_1, \ldots, \vec{w}_n$ not linearly independent

A contradiction

Exchange $\vec{w}_1$ and $\vec{w}_i$'s $n < m$ is also impossible

Therefore $n = m$. $\square$
For a vector space
Any basis has the same # of vectors, call it the dim'

Definition
The dimension of a vector space is the number of vectors in any basis
Example:

\[ V = \{ \text{2x2 matrices} \} \]

Check

\[ \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \} \]

is a basis.

So \( \dim V = 4 \).

Question: What is the \( \dim \) of the space of all \( m \times n \) matrices?
Example

The solution of

$$\frac{d^2 y}{dx^2} = -y$$

We have mentioned that the space of all functions $f(x)$ is a vector space. The solution of $\frac{d^2 y}{dx^2} = -y$ $W$ is a subspace.
Check

1) \[ \frac{d^2 y_1}{dx^2} = -y_1 \]  
\[ \frac{d^2 y_2}{dx^2} = -y_2 \]  

(0 + 2) \[ \frac{d^2 (y_1 + y_2)}{dx^2} = -(y_1 + y_2) \]

2) \( c \in \mathbb{R} \). \[ \frac{d^2 y}{dx^2} = -y \]
\[ \frac{d^2 (cy)}{dx^2} = c \frac{d^2 y}{dx^2} = c (-y) \]
\[ = -cy \]

So \( W \) is a subspace, thus a vector space.
All the solutions to \( \frac{d^2y}{dx^2} = -y \)
are of the form
\[ y = c \sin x + d \cos x. \]
\[ \{ \sin x, \cos x \} \text{ is a basis for } W \]
\[ \dim W = 2. \]
Facts:

1. Any linearly independent subset can be extended to a basis.

2. If a subset $S$ spans $V$, then we can remove some vectors from $S$ and get a basis.

Corollary:

$$\dim V = n.$$ 

1. If a linearly independent subset $S$ has $n$ elements, then it is a basis.

2. If a subset $S$ of $n$ elements spans $V$, then it is a basis.
Questions:

Given a subspace $W$ of $\mathbb{R}^n$.

1) What is the dimension of $W$?

2) How to get a basis of $W$?

3) Given a vector $\vec{v} \in \mathbb{R}^n$, how to know if $\vec{v}$ is in $W$?

4) Given a vector $\vec{u}$ in $W$, how to express $\vec{v}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ from 2)?