Here is the instruction for the Final Exam:

1. This exam has three parts. The exam time is 12 - 2 pm.

2. Part I has 5 shorter questions about concepts. (6 points each).

3. Part II has 5 (True/False) questions (4 points each). If the answer is false, you need to Either provide a counter example, OR to explain what condition is missing. Otherwise you only get 2 points even if the correct answer is false.

4. Part III has 5 standard problems. (10 points each).

5. Turn off and put away your cell phone.


7. Read each question carefully, and answer each question completely.

8. Show all of your work; no credit will be given for unsupported answers.

9. Write all your solutions clearly and legibly; no credit will be given for illegible solutions.

10. If any question is not clear, ask for clarification.

11. A vector $(a, b, c)$ means a column vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
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The scope of the final exam will be everything we learnt this semester. From the very
beginning to SVD (not including pseudo inverse). Let me divide the guide into before Mid-
term 2 and after Mid-term 2.

1 Before (and include) Mid-term 2

The problems will be similar to the mid-term exams. The problems in Part I and II will
involve understanding concepts like linear spaces, linear subspaces, linearly independence,
ranks, dimensions, 4 subspaces \( C(A), C(A^T), N(A), N(A^T) \) and their relations, orthogonal
and orthogonal complements, all equivalent statements about invertibility of a matrix(or
linear map).

The problems in Part III will test skills like Gaussian elimination, LU factorization, solv-
ing linear equations, orthogonal projections, Least square approximations, Gram-Schmidt,
computing determinants. Of course due to the limit of time, we are not going to test all
these. But everything is included in the 2 mid-term exams. So I don’t give extra samples.

2 After Mid-term 2

The exam will focus more on the materials after mid-term 2. The main material you can
study is the homework problems. In addition, I provide some sample problems here.

2.1 Part I and Part II

We test concepts like:

1. Linear maps, the kernel and image.

2. Eigenvalues and eigenvectors.

3. You need to know how to see if a matrix is diagonalizable or not.

4. All equivalent statements about orthogonal/unit matrices.

5. All the properties of symmetric/Hermitian matrices. In particular, you need to be
familiar with the argument used in the proof of all eigenvalues of symmetric/Hermitian
matrices are real and eigenvectors with distinct eigenvalues for symmetric/Hermitian
matrices are orthogonal. You will be asked to use similar arguments to prove similar
problems.

6. All the equivalent ways of saying positive definite.
Sample Problems:

Problem 1. Let $T : V \to W$ be a linear transformation from a 3-dimensional vector space $V$ to a 2-dimensional vector space $W$. If the image of $T$ is not the whole $W$, what are the possible dimensions of $\ker T$?

Solution 1. Since the image of $T$ is not the whole $W$, the possible rank of $T$ are 0 or 1. By nullity + rank = dim $V = 3$, the possible dimension of $\ker T$ is 3 or 2.

Problem 2. Show that all eigenvalues of a real skew-symmetric matrix are pure imaginary. [Note the eigenvector for a skew-symmetric matrix is usually a complex vector.]

Solution 2. If $A$ is real and symmetric, then $A^H = A^T = -A$. Suppose $\lambda$ is an eigenvalue for $A$. That means there is a nonzero $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v} \quad (1)$$

Take $^H$ we have

$$\vec{v}^H A^H = \bar{\lambda}\vec{v}^H$$

That means

$$\vec{v}^H A = -\bar{\lambda}\vec{v}^H \quad (2)$$

Compare $\vec{v}^H$ times (1) with (2) times $\vec{v}$

$$\vec{v}^H A\vec{v} = \lambda\vec{v}^H \vec{v}$$

$$\vec{v}^H A\vec{v} = -\bar{\lambda}\vec{v}^H \vec{v}$$

Therefore

$$\lambda\vec{v}^H \vec{v} = -\bar{\lambda}\vec{v}^H \vec{v}$$

But $\vec{v}^H \vec{v} = ||\vec{v}||^2 \neq 0$. That implies $\lambda = -\bar{\lambda}$, i.e. $\lambda$ is pure imaginary.

Problem 3. Let $S$ be an $n \times n$ real symmetric matrix and $\vec{v}$ an eigenvector for $S$ with eigenvalue $\lambda$. Let $W = \text{span}\{\vec{v}\}$.

a) If $\vec{w} \in W$, show that $S\vec{w} \in W$.

b) If $\vec{u} \in W^\perp$, show that $S\vec{u} \in W^\perp$.

Solution 3. $\vec{v}$ is an eigenvector for $S$ with eigenvalue $\lambda$ means

$$S\vec{v} = \lambda\vec{v}.$$  

a) Since $W = \text{span}\{\vec{v}\}$, $\vec{w} = c\vec{v}$ for some scalar $c$. Then $S\vec{w} = S(c\vec{v}) = cS\vec{v} = c\lambda\vec{v}$ is still in $W$. b) Since $\vec{u} \in W^\perp$, $\vec{v}^T \vec{u} = 0$. Then $\vec{v}^T(S\vec{u}) = \vec{v}^T S\vec{u} = (S^T\vec{v})^T \vec{u} = (S\vec{v})^T \vec{u} = \lambda\vec{v}^T \vec{u} = 0$. Therefore $S\vec{u} \in W^\perp$. 

Problem 4. (True/False) Suppose that $\vec{u}, \vec{v}, \vec{w}$ are vectors in a vector space $V$ and $T : V \to W$ is a linear map. If $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then $T(\vec{u}), T(\vec{v}), T(\vec{w})$ are also linearly dependent.

Solution 4. It is true. The reason (You are not required to write this if it is true) is that if $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, there is a nontrivial linear relation

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$$

for $a, b, c$ not all zero.

Therefore

$$aT(\vec{u}) + bT(\vec{v}) + cT(\vec{w}) = T(a\vec{u} + b\vec{v} + c\vec{w}) = T(\vec{0}) = \vec{0}.$$ for $a, b, c$ not all zero.

That means $T(\vec{u}), T(\vec{v}), T(\vec{w})$ are also linearly dependent.

Problem 5. (True/False) If $A$ is diagonalizable square matrix, then so is $A^3$.

Solution 5. True. (Again if it is true, you don’t need to explain the reason) $A$ is diagonalizable means $A = P\Lambda P^{-1}$ for some diagonal matrix $\Lambda$. So $A^3 = P\Lambda P^{-1}P\Lambda P^{-1}P\Lambda P^{-1} = P\Lambda^3 P^{-1}$ and $\Lambda^3$ is a diagonal matrix. Therefore $A^3$ is also diagonalizable.

2.2 Part III

This part will test skills:

1. Solving for eigenvalues/eigenvectors.

2. diagonalizing a square matrix if it is diagonalizable.

3. test if a symmetric/Hermitian matrix is positive definite.

4. In some cases you can tell if a matrix is diagonalizable without computing eigenvectors. For example, an $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable. Symmetric/Hermitian/ orthogonal/ unitary/ skew-symmetric/ skew-Hermitian matrices are diagonalizable. In particular symmetric matrices can be diagonalized by orthogonal matrices $S = Q\Lambda Q^T$. Hermitian matrices can be diagonalized by unitary matrices $H = U\Lambda U^H$. For orthogonal/skew-symmetric matrices, even they are real, their eigenvectors are in general complex vectors.

5. Computing $S = Q\Lambda Q^T$ and $H = U\Lambda U^H$. This is equivalent to diagonalizing $S$ and $H$.

6. 5) is also equivalent to looking for orthonormal basis for an inner product

7. compute SVD.
Sample Problems:

**Problem 6.** Which matrices can be diagonalized? Which can not be diagonalized? Note that you don’t need to compute the eigenvectors for any of the case.

1. \( A = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \end{pmatrix} \)

2. \( B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \)

**Solution 6.** \( A \) is diagonalizable because it is orthogonal. \( B \) is not diagonalizable. To show that let’s assume what if \( B \) is diagonalizable and get a contradiction. Note all the eigenvalues of \( B \) are 3. If \( B \) is diagonalizable, then

\[ B = P \Lambda P^{-1} \text{ for } \Lambda = 3I. \]

Then \( B = 3PP^{-1} = 3P \), a contradiction.

**Problem 7.** Find the new basis in \( \mathbb{C}^3 \) that diagonalize the above \( A \). That is find \( P \) such that \( P^{-1}AP = \Lambda \) such that \( \Lambda \) is diagonal.

**Solution 7.** The characteristic polynomial for \( A \) is

\[ p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & \lambda - 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & \lambda \end{vmatrix} = \lambda^3 - 1/\sqrt{2}\lambda^2 - 1/\sqrt{2}\lambda + 1 \]

The three roots are \(-1, 1 + \sqrt{2} + \sqrt{5 - 2\sqrt{2}}i\). For the convenience of typing, let’s denote

\[ \alpha = 1 + \sqrt{2} + \sqrt{5 - 2\sqrt{2}}i, \quad \bar{\alpha} = 1 + \sqrt{2} - \sqrt{5 - 2\sqrt{2}}i \]

Then if \( \lambda = -1 \), one eigenvector is \((-1, \sqrt{2} - 1, 1)\). If \( \lambda = \alpha \), one eigenvector is \((\alpha, \sqrt{2}\alpha^2 - 1, 1)\). If \( \lambda = \bar{\alpha} \), one eigenvector is \((\bar{\alpha}, \sqrt{2}\bar{\alpha}^2 - 1, 1)\). One possible answer (the answer is not unique)

\[ P = \begin{bmatrix} -1 & \alpha & \bar{\alpha} \\ \sqrt{2} - 1 & \sqrt{2}\alpha^2 - 1 & \sqrt{2}\bar{\alpha}^2 - 1 \\ 1 & 1 & 1 \end{bmatrix} \]

**Problem 8.** Find an orthogonal matrix \( Q \) that diagonalize \( S = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \).
Solution 8. The characteristic polynomial for $S$ is

$$p(\lambda) = \det(\lambda I - S) = \begin{vmatrix} \lambda - 1 & 1 & 0 \\ 1 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = \lambda(\lambda - 2)^2$$

If $\lambda = 0$ a normal eigenvector is $(1/\sqrt{2}, 1/\sqrt{2}, 0)$. If $\lambda = 2$, take two orthonormal ones $(1/\sqrt{2}, -1/\sqrt{2}, 0)$ and $(0, 0, 1)$. Therefore

$$Q = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}$$

is actually the $A$ in the above problem.

Problem 9. Find the SVD of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. That is, write $A$ as $\sum_i \sigma_i \bar{u}_i \bar{v}_i^T$.

Solution 9. Compute

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Choose orthonormal eigenvectors

$$u_1 = (1, 0), u_2 = (0, 1)$$

$$\sigma_1 = \sigma_2 = \sqrt{2}$$

Next we compute $v_1, v_2$

$$v_1 = \frac{A^T u_1}{\sigma_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = \frac{A^T u_2}{\sigma_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The SVD is

$$A = \sqrt{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} + \sqrt{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(3)

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(4)

Either (3) or (4) is correct.