Math 312: Problem Set 8

Due date: Nov 9, Mon. Turn in homework in class.

1. Let $T$ be a linear operator on $\mathbb{R}^3$ which is represented relative to the standard basis by the matrix

$$
\begin{pmatrix}
-9 & 4 & 4 \\
-8 & 3 & 4 \\
-16 & 8 & 7
\end{pmatrix}
$$

a) Find out all eigenvalues of $T$.

b) Find a basis consisting of eigenvectors.

c) Show that $T$ can be diagonalizable.

2. Let $A$ be an arbitrary $3 \times 3$ matrix over the field of complex numbers. We form the matrix $xI - A$ with polynomial entries, the $i,j$ entry of this matrix being the polynomial $\delta_{ij}x - A_{ij}$. Here $\delta_{ij}$ is 1 if $i = j$, and otherwise 0. The characteristic polynomial is $f = \det(xI - A)$. Show that $f$ is a monic polynomial of degree 3. (I have proved this in class. This exercise asks you to write a complete proof for $3 \times 3$ case.)

3. Suppose $A$ is a $2 \times 2$ matrix

$$
\begin{pmatrix}
2 & 3 \\
-1 & 1
\end{pmatrix}
$$

a) Let $T$ be the linear operator on $\mathbb{R}^2$ represented by the matrix $A$ relative to the standard basis of $\mathbb{R}^2$. Find all eigenvalues of $T$. Is $T$ diagonalizable?

b) Let $T_{\mathbb{C}}$ be the linear operator on $\mathbb{C}^2$ represented by the matrix $A$ relative to the standard basis of $\mathbb{C}^2$. Find all eigenvalues of $T_{\mathbb{C}}$. Is $T_{\mathbb{C}}$ diagonalizable? (Note the fields are different in a) and b).)

4. Let $T$ be a linear operator on the $n$-dimensional vector space $V$, and suppose that $T$ has $n$ distinct eigenvalues. Prove that $T$ is diagonalizable.

5. Say a square matrix $A$ has the property that $A^3 - A = 0$. What are the possible eigenvalues of $A$? Justify your answer.
6. Consider $M = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$ as a matrix over real numbers $\mathbb{R}$. For which real numbers $a$ and $b$ can $M$ be diagonalized? Justify your response.

7. Let $A$ and $B$ be $n \times n$ matrices over the field $F$. Prove that if $(I - AB)$ is invertible, then $(I - BA)$ is also invertible and the inverse is 

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A$$

8. Use the problem above to prove that, if $A$ and $B$ are $n \times n$ matrices over the field $F$, then $AB$ and $BA$ have precisely the same eigenvalues in $F$. (Hint. Use the above theorem, you can show that if $\lambda$ is not an eigenvalue of $AB$, then it is not an eigenvalue of $BA$ either, and vice versa.)