Math 312: Exam 1

Sep 30, 2015

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
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<td>3</td>
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<td>Total</td>
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1
Let \[ A = \begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix}. \]

Use elementary row operations to discover whether it is invertible, and to find the inverse in case it is. (You should use elementary row operations. If you use determinants, you don’t get points.)
The matrix

\[ B = \begin{pmatrix} 1 & 1 + i & i \\ 0 & 1 - i & i \\ -i & 1 & i \end{pmatrix} \]

is an invertible matrix.

a) Find the inverse of \( B \) by using elementary row operations.

b) Since \( B \) is invertible, we know that the column vectors of \( B \) is a basis of \( \mathbb{C}^3 \). Note that we can view elements in \( \mathbb{C}^3 \) either as column vectors or row vectors. Denote this new basis by \( \mathcal{B} \).

\[ \mathcal{B} = \left\{ \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 + i \\ 1 - i \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} i \\ i \\ i \end{pmatrix} \right\} \]

For a vector in \( \mathbb{C}^3 \)

\[ \alpha = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \]

what are the coordinates of \( \alpha \) with respect to the basis \( \mathcal{B} \)?
Consider a $3 \times 3$ matrix

$$A = \begin{pmatrix} a & c & g \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix}.$$ 

We first study the solutions of a system of linear equations $AX = 0$. Here $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is a column vector. We want to show that if $abc = 0$, then $A$ is not invertible.

a) If $c = 0$, prove that there is a nontrivial solution for $AX = 0$ by using a theorem we learnt in class. (Hint: think about the number of equations and number of unknowns.)

b) Now we can assume that $c \neq 0$. If $b = 0$, prove that there is a nontrivial solution for $AX = 0$.

c) If $a = 0$, find out a nontrivial solution.

d) Prove that if $abc = 0$, then the matrix $A$ is NOT invertible.
We continue to study the $3 \times 3$ matrix in Problem 3

\[
A = \begin{pmatrix}
a & e & g \\
0 & b & f \\
0 & 0 & c
\end{pmatrix}.
\]

We want to show if $abc \neq 0$, then $A$ is invertible.

e) Assume $abc \neq 0$. Find a row-reduced echelon matrix $R$ that is row equivalent to $A$.

f) Prove that if $abc \neq 0$, then $A$ is invertible. (Use a theorem we learnt in class.)

g) Combine with the conclusions we get from Problem 3, prove that the matrix $A$ is invertible, if and only if $abc \neq 0$. Note that “if and only if” means you need to consider both directions.
If a square matrix $M$ has the property that $M^4 - M^2 + 2M - I = 0$, prove that $M$ is invertible. Here $I$ is the identity matrix of the same size. [Hint: find a matrix $N$ such that $NM = I$.]