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1.1 Fields

Example: R, C.

A field means we have two different operations

multiplication ×

addition +

They satisfy

1) $x + y = y + x$

2) $x + (y + z) = (x + y) + z$

3) There exists 0 in F such that
   $x + 0 = x$ for any $x$ in F.

4) For any $x$ in F, there is $-x$
   such that $x + (-x) = 0$
5) \( xy = yx \)
6) \( x(yz) = (xy)z \)
7) There exists \( 1 \neq 0 \) in \( F \), such that \( x \cdot 1 = x \)
8) For any \( x \neq 0 \), there exists \( x^{-1} \in F \), such that \( x(x^{-1}) = 1 \)
9) \( x(yz) = xy + xz \)

**Definition:** A **field** \( F \) is a set with two operations such that \( 1) \) - \( 9) \) are satisfied.

**NOT Example:** \( \mathbb{N} = \) the set of natural numbers
\[ \mathbb{Z} = \) the set of integers \]
There are fields that might be unfamiliar to you.

\[ 1 + 1 + 1 \cdots + 1 = 0 \]

\[ \underbrace{p \text{ times}} \]

For \( p \) a prime number,

\[ \mathbb{F}_p = \{ \text{congruence class modulo } p \} \]

\[ = \{ 0, 1, 2, \ldots, p-1 \} \]

\[ \overline{a} + \overline{b} = \overline{a+b} \pmod{p} \]

\[ \overline{a} \cdot \overline{b} = \overline{a \cdot b} \pmod{p} \]

and \( \overline{p} = 0 \) in \( \mathbb{F}_p \)!
1.2 System of linear equations

Fix $F$ a field.
(think about $\mathbb{R}$ or $\mathbb{C}$)

$$\begin{cases} A_{11} x_1 + \cdots + A_{1n} x_n = y_1 \\ \vdots \\ A_{m1} x_1 + \cdots + A_{mn} x_n = y_m \end{cases}$$

Here $A_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$

$y_1, \ldots, y_m$

are all elements in $F$.

$x_1, x_2, \ldots, x_n$ are unknowns.

We want to solve for

$$(x_1, x_2, \ldots, x_n)$$
Definition

If \( y_1 = y_2 = \ldots = y_m = 0 \)

The system is called homogenous

Example:

\[
\begin{align*}
\begin{cases}
    x_1 + x_2 &= 0 \\
    x_2 + 3x_3 &= 0
\end{cases}
\end{align*}
\]

\( n = 3 \)

\( m = 2 \)

Express both \( x_1, x_3 \) in terms of \( x_2 \).

\( x_1 = -x_2 \)

\( x_3 = -\frac{x_2}{3} \)

\( x_1 \) can be any element \( a \) in \( F \)

\( (x_1/x_2/x_3) = (-a, a, -\frac{a}{3}) \)
Example:

\[
\begin{align*}
2x_1 - x_2 + x_3 &= 0 \quad (1) \\
-x_1 + 3x_2 + 4x_3 &= 0 \quad (2)
\end{align*}
\]

Eliminate some unknowns

Multiply the second one by 2

\[
2x_1 + 6x_2 + 8x_3 = 0 \quad (3)
\]

Subtract (3) from (1)

\[-7x_2 - 7x_3 = 0\]

\[x_2 + x_3 = 0\]

\[x_3 = -x_2\]

Substitute \(x_3\) in (2)

\[x_1 + 3x_2 + 4(-x_2) = 0\]

\[x_1 = x_2\]

\((a, a, -a)\) is a solution