Aug 31  Lecture 2
Recall the example
\[ \begin{align*}
&2x_1 - x_2 + x_3 = 0 \\
&x_1 + 3x_2 + 4x_3 = 0
\end{align*} \]

Why can we solve an equation this way?
The solutions for \( x \) are also the solutions for
\[ \begin{align*}
&x_1 + x_3 = 0 \\
&x_1 + x_3 = 0
\end{align*} \]

This is because \( x_2 + x_3 \) is a linear combination of equations in \( x \).
\[ x_2 + x_3 = -\frac{1}{7} \left( 2x_1 - x_2 + x_3 \right) \\
\quad + \frac{2}{7} \left( x_1 + 3x_2 + 4x_3 \right) \]

\[ x_1 + x_3 = \frac{3}{7} \left( 2x_1 - x_2 + x_3 \right) \\
\quad + \frac{1}{7} \left( x_1 + 3x_2 + 4x_3 \right) \]

In general,

\[
\begin{aligned}
A_{11} x_1 + \cdots + A_{1n} x_n &= y_1 \\
\vdots &\\
A_{m1} x_1 + \cdots + A_{mn} x_n &= y_m
\end{aligned}
\]

If we have another equation

\[
\begin{aligned}
c_1 (A_{11} x_1 + \cdots + A_{1n} x_n) + \cdots \\
+ c_m (A_{m1} x_1 + \cdots + A_{mn} x_n)
\end{aligned}
\]

\[ = c_1 y_1 + \cdots + c_m y_m \]

then any solution of \( \star \) is also a solution of
If
\[ x \neq 0 \]
\[
\begin{align*}
B_1 x_1 + \ldots + B_{1n} x_n &= z_1 \\
B_{k1} x_1 + \ldots + B_{kn} x_n &= z_k
\end{align*}
\]
Each
\[
B_1 x_1 + \ldots + B_{kn} x_n = z_i
\]
\[ 1 \leq i \leq k \]
is a linear combination of equations in \( x \).

Then any solution of \( x \) is a solution of \( xx \).

If on the other hand, each equation in \( x \)
\[
A_{11} x_1 + \ldots + A_{1n} x_n = y_i
\]
\[ 1 \leq i \leq m \]
is also a linear combination of equations in **
then any solution of ** is also a solution of *.

In this case (= both hold) *
and ** have the same solutions.

To summarize this condition,
we call * and ** are equivalent.
Definition.
We have two systems of linear equations \( \star \) and \( \star \star \).
They are called equivalent, if each equation in \( \star \) is a linear combination of equations in \( \star \star \) and vice versa.

Theorem:
Equivalent system of linear equations have exactly the same equations.
1.3 Matrices and Elementary Row Operations

How to solve a system of linear equations?

Find an equivalent system of linear equations that is of a very simple form.

To do it more effectively, we use matrices.
\[
\begin{align*}
\begin{cases}
A_{11} x_1 + \ldots + A_{1n} x_n &= y_1 \\
\quad & \quad \\
A_{m1} x_1 + \ldots + A_{mn} x_n &= y_m
\end{cases}
\end{align*}
\]

Rewrite it as

\[
A \bar{x} = \bar{y}
\]

for

\[
A = \begin{pmatrix}
A_{11} & \ldots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{m1} & \ldots & A_{mn}
\end{pmatrix}_{m \times n}
\]

\[
\bar{x} = \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}_{n \times 1 \text{ matrix}}
\]

\[
\bar{y} = \begin{pmatrix}
y_1 \\
\vdots \\
y_m
\end{pmatrix}_{m \times 1 \text{ matrix}}
\]
For any $m \times n$ matrix $A$ we apply 3 operations called elementary row operations.

1. Multiply one row of $A$ by a non-zero number $C \in F$.

2. Replace row $r$ by row $r$ plus $C$ times row $s$, $s \neq r$.

3. Interchange two different rows.

Regard $A$ as an $m \times n$ matrix as $m$ rows of vectors.
These operations are invertible:

\[ A \rightarrow A' \rightarrow A \]

\[ \text{row } r \times c \]

\[ A \rightarrow A' \rightarrow A \]

\[ \text{row } r - c \times \text{rows} \]

\[ A \rightarrow A' \rightarrow A \]

(3)

Definition:

A and B are m x n matrices over F. A and B are row-equivalent, if B is obtained from A by a finite sequence of elementary row operations.
Remark.

In this case, $A$ is also obtained from $B$ by a finite sequence of elementary row operations. That's why I say $A$ and $B$ are equivalent. This is an equivalence relation.
**Theorem:**

If $A$ and $B$ are row-equivalent matrices, then the corresponding homogeneous system of linear equations

$$A\overline{x} = 0 \quad B\overline{x} = 0$$

are equivalent.

**Why?** We only need to check for each elementary row operation, $A_k \rightarrow A_{k+1}$.

The rows of $A_k$ are linear combination of rows of $A_{k+1}$ and vice versa.