Dec 4

Suppose $V, W$ are inner product spaces over $F$.

Definition A (linear map $T : V \rightarrow W$ is called an \underline{(inner product) isomorphism} if:

i) $T$ is an isomorphism as a linear map.

ii) $(T\alpha, T\beta) = (\alpha, \beta)$ for any $\alpha, \beta \in V$

Definition Two inner product spaces $V$ and $W$ are \underline{isomorphic (as inner product spaces)} if there exists an \underline{(inner product) isomorphism} $T : V \rightarrow W$. 
\[ \mathbb{R}^2 \to \mathbb{R}^2 \]

rotation by \( \theta \)

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

is an isomorphism. It preserves the metric (or distance)

scaling is not \( (\begin{pmatrix}
1 & 0 \\
0 & 4
\end{pmatrix}) \) an isometry.
Theorem: V, W inner product spaces over F dim V = dim W.
T: V → W The following are equivalent.

i) T is an inner product isomorphism

ii) ||Tα|| = ||α|| for any α ∈ V

iii) T carries every orthonormal basis for V to an orthonormal basis for W

iv) T carries some orthonormal basis for V to an orthonormal basis for W.
Proof : i) \implies ii)

Since \((Tx, Tx) = (x, x)\)

ii) \implies iii)

First if \(Tx = 0\)

\(\|x\| = \|Tx\| = 0\) I don't need this

\(\implies x = 0\)

\(T\) is nonsingular, therefore \(T\) carries a basis to a basis

Now suppose \(B = \{x_1, \ldots, x_n\}\) is an orthonormal basis for \(V\).

\(\|Tx_i\| = \|x_i\| = 1\)

for each \(i\)

Suppose \(k \neq l\)

\((x_k, x_l) = 0\)

\((x_k, x_l) = 0 \iff F = \mathbb{C}\)
Therefore
\[ \| \alpha_k + \alpha_\ell \|^2 = \| \alpha_k \|^2 + \| \alpha_\ell \|^2 \]
\[ \left( \| \alpha_k + i \alpha_\ell \|^2 = \| \alpha_k \|^2 + \| \alpha_\ell \|^2 \right) \]
\[ \Rightarrow \]
\[ \| T(\alpha_k + \alpha_\ell) \|^2 = \| T \alpha_k \|^2 + \| T \alpha_\ell \|^2 \]
\[ \left( \| T(\alpha_k + i \alpha_\ell) \|^2 = \| T \alpha_k \|^2 + \| T i \alpha_\ell \|^2 \right) \]
However
\[ \| T(\alpha_k) + T(\alpha_\ell) \|^2 = \| T \alpha_k \|^2 \]
\[ + \| T(\alpha_\ell) \|^2 + 2 \text{Re} \left( T \alpha_k, T \alpha_\ell \right) \]
\[ \left( \| T(i \alpha_\ell) \|^2 + \cdots + 2 \text{Re} \left( T \alpha_k, i T \alpha_\ell \right) \right) \]
\[ \Rightarrow \quad \text{Re} \left( T_{ak}, T_{dl} \right) = 0 \]

\[
\begin{align*}
\text{Re} \left( T_{ak}, i T_{dl} \right) &= 0 \quad F = C \\
\text{If } F = \mathbb{R} &,
\end{align*}
\]

\[ \text{This means } \left( T_{ak}, T_{dl} \right) = 0 \]

\[ T_{ak} \perp T_{dl} \]

\[ \text{If } F = \mathbb{C} . \]

\[ \text{We have} \]

\[ \text{Re} \left( T_{ak}, T_{dl} \right) = 0 \]

\[ \text{Re} \ i \left( T_{ak}, T_{dl} \right) = 2 \text{m} \left( T_{ak}, T_{dl} \right) = 0 \]

\[ \Rightarrow \quad \left( T_{ak}, T_{dl} \right) = 0 \]

\[ T_{ak} \perp T_{dl} . \]

\[ \{ T_{a_1}, \ldots, T_{a_n} \} \text{ also orthonormal} \]
\{T\alpha_1, \ldots, T\alpha_n\} orthogonal and nonzero \Rightarrow
\{T\alpha_1, \ldots, T\alpha_n\} linearly independent, n = \dim V = \dim W
\Rightarrow \{T\alpha_1, \ldots, T\alpha_n\} is a basis

iii \Rightarrow iv) Obvious

iv) \Rightarrow i). Suppose \{\alpha_1, \ldots, \alpha_n\}
an orthonormal basis for V.
\{T\alpha_1, \ldots, T\alpha_n\} an orthonormal basis for W.

T carries a basis to a basis
\Rightarrow T is nonsingular, thus an isomorphism.
\((Ta_i, Ta_j) = \delta_{ij} = (\alpha_i, \alpha_j)\)

For any \(\alpha = x_1a_1 + \cdots + x_na_n\)
\[\beta = y_1a_1 + \cdots + y_na_n\]
\(T\alpha = x_1T\alpha_1 + \cdots + x_nT\alpha_n\)
\(T\beta = y_1T\alpha_1 + \cdots + y_nT\alpha_n\)

We have shown that relative to orthonormal basis
\((\cdot, \cdot) = \text{the dot product}\)

Therefore
\[(T\alpha, T\beta) = \sum_{1}^{n} x_i y_i\]
\[= (\alpha, \beta) \quad \square\]
We focus on $V = W$

$T : V \rightarrow V$ linear operator on an inner product space $V$.

**Theorem** $T$ is an isomorphism if and only if $T$ invertible and $T^* = T^{-1}$

**Proof:** "only if."

Suppose $T$ is an isomorphism

$$( \alpha, T^*T\beta ) = ( T\alpha, T\beta ) = ( \alpha, \beta )$$

For any $\alpha, \beta$

$\Rightarrow$ $T^*T\beta = \beta$ for any $\beta$

$\Rightarrow$ $T^*T = I \Rightarrow T^* = T^{-1}$

"if" $T^* = T^{-1}$

$$(T\alpha, T\beta) = (\alpha, T^*T\beta) = (\alpha, \beta) \square$$
$F = \mathbb{R}, \quad T: V \rightarrow V$ isomorphism is called **orthogonal**

Since $A^* = A^T$

A square matrix $A$ is called **orthogonal** if $A^TA = I$

$F = \mathbb{C}, \quad T: V \rightarrow V$ isomorphism is called **unitary**

A square matrix $A$ is called **unitary** if $A^TA = I$
Theorem:

\( F = \mathbb{R} \). A square matrix \( A \) is orthogonal if and only if the columns form an orthonormal basis. if and only if the rows form an orthonormal basis.

\( F = \mathbb{C} \). A square matrix \( A \) is unitary if and only if the columns form an orthonormal basis if and only if the rows form an orthonormal basis.

Proof: A maps the standard basis of \( F^n \) to the columns.
By the theorem
A is an isomorphism if and only if these columns again form an orthonormal basis.
Also, A orthogonal (unitary)  
(\iff) A^T orthogonal (unitary)

That's how I knew
U_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ is orthogonal.}
Definition,
A, B square matrices
A, B are called unitarily equivalent if \( B = P^{-1}AP \)
for \( P \) unitary.
A, B are called orthogonally equivalent, if \( B = P^TAP \)
for \( P \) orthogonal.
Now $A$ is unitarily equivalent to a diagonal matrix

$p^{-1}AP$ is diagonal

find a basis of eigenvectors

Make sure $P$ is unitary

$\Leftrightarrow$ The columns of $P$

form an orthonormal basis.

If $p^{-1}AP$ is diagonal for

$P$ orthogonal  The same

$P$ orthogonal $\Leftrightarrow P^T = P^{-1}$
Find a basis of eigenvectors $I$ make sure this basis is orthonormal

The columns of $P$ form an orthonormal basis