Nov 11

Direct sum. Blocked matrices.

\[ T: V \rightarrow V. \]

\( W \) is a \( T \)-invariant subspace

\[ Tw: W \rightarrow W \] the restriction of \( T \) on \( W \).

We first choose a basis of \( W \) \( \{x_1, \ldots, x_r\} \)

\[ \Rightarrow \{x_1, \ldots, x_r\} \text{ linearly independent. Extend } \{x_1, \ldots, x_r\} \]

to a basis of \( V \).
\[ \beta = \{ \alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n \} \]

basis of \( W \)

Then if \( 1 \leq i \leq r \)

\( \alpha_i \) in \( W \)

\( T \alpha_i = T_w \alpha_i = \) linear combination of \( \{ \alpha_1, \ldots, \alpha_r \} \)

The matrix of \( T \) relative to \( \beta = \{ \alpha_1, \ldots, \alpha_n \} \) is in the form.

\[
\begin{pmatrix}
A & B \\
O & C
\end{pmatrix}
\]

\( A \) is the matrix of \( T_w \) relative to \( \{ \alpha_1, \ldots, \alpha_r \} \)
A, r x r matrix, is the matrix of \textbf{T} relative to \{x_1, \ldots, x_r\}

What if we have two 
\textbf{T}-invariant subspaces \textbf{W}_1 and \textbf{W}_2, and the basis of 
\textbf{W}_1 plus the basis of \textbf{W}_2 is exactly a basis of \textbf{V}?

\textbf{We need to formalize this precisely.}
Definition.

Let \( W_1, \ldots, W_k \) be subspaces of the vector space \( V \). We say that \( W_1, \ldots, W_k \) are linearly independent if

\[
d_1 + \ldots + d_k = 0 \quad d_i \in W_i
\]

implies that \( d_i = 0 \) for each \( i \).

Example: We have proved eigenspaces with different eigenvalues are linear independent.
Lemma. Let $V$ be a finite-dimensional vector space, $W_1$ and $W_2$ be two subspaces that span $V$. The following are equivalent:

a) $W_1$ and $W_2$ are linearly independent.

b) $W_1 \cap W_2 = \{0\}$

c) If $\{\alpha_1, \ldots, \alpha_r\}$ is a basis for $W_1$, $\{\beta_1, \ldots, \beta_s\}$ is a basis for $W_2$, then $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s\}$ is a basis for $V$. 
Proof  a) $\Rightarrow$ b)

Suppose $W_1$ and $W_2$ are linearly independent.

For any $x \in W_1 \cap W_2$

Then $x \in W_1$, $-x \in W_2$ and

$$x + (-x) = 0$$

Since $W_1$ and $W_2$ are linearly independent $x = 0$.

b) $\Rightarrow$ c) We want to show

$\{x_1, \ldots, x_r, \beta_1, \ldots, \beta_s\}$ is linearly independent.

For any linear relation

$$a_1 x_1 + \ldots + a_r x_r + b_1 \beta_1 + \ldots + b_s \beta_s = 0$$
Let \( \alpha = a_1 \alpha_1 + \ldots + a_r \alpha_r \)
\( \beta = b_1 \beta_1 + \ldots + b_r \beta_r \)
\( \alpha \) in \( W_1 \), and \( \beta \) in \( W_2 \)

\[ 0 = \alpha + \beta \]
\( \alpha = -\beta \) is also in \( W_2 \)
\( \beta = -\alpha \) is also in \( W_1 \)

Therefore \( \alpha \in W_1 \cap W_2 \)
\( \beta \in W_1 \cap W_2 \)

But \( W_1 \cap W_2 = \{0\} \)

\( \implies \alpha = 0 \), \( \beta = 0 \)

\( a_1 \alpha_1 + \ldots + a_r \alpha_r = 0 \)
\( b_1 \beta_1 + \ldots + b_s \beta_s = 0 \)

But \( \{ \alpha_1, \ldots, \alpha_r \} \) basis for \( W_1 \)
\( \{ \beta_1, \ldots, \beta_s \} \) basis for \( W_2 \)
\[ \Rightarrow \quad a_1 = \cdots = a_r = 0 \]
\[ \quad b_1 = \cdots = b_s = 0 \]
\[ c) \Rightarrow a) \]
For any \( \alpha \in W_1, \beta \in W_2 \)
and \( \alpha + \beta = 0 \)
\[ \alpha = a_1 \alpha_1 + \cdots + a_r \alpha_r \]
\[ \beta = b_1 \beta_1 + \cdots + b_s \beta_s \]
and \( a_1 \alpha_1 + \cdots + a_r \alpha_r + b_1 \beta_1 + \cdots + b_s \beta_s = 0 \)

Since \( \{ \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s \} \) is
a basis for \( V \).
\[ a_1 = \cdots = a_r = b_1 = \cdots = b_s = 0 \]
\[ \Rightarrow \quad \alpha = 0 \quad \beta = 0 \]
\( W_1 \) and \( W_2 \) are linearly independent. \( \square \)
In the above situation, i.e.
\[ V = W_1 + W_2 \]
and \( W_1, W_2 \) linearly independent
We say \( V \) is a direct sum
of \( W_1 \) and \( W_2 \) denote by
\[ V = W_1 \oplus W_2. \]

Example:
\[ \mathbb{R}^2 \text{ is a direct sum of} \]
\[ W_1 = \{ (x,0) \} \cong \mathbb{R} \]
and \( W_2 = \{ (0,y) \} \cong \mathbb{R}. \)

First any \((x,y) \in \mathbb{R}^2\)
can be written as \((x,0)+(0,y)\)
\[ \mathbb{R}^2 = W_1 + W_2. \]
Secondly \( W_1 \cap W_2 = \{ (0,0) \} \)
Example: $T: V \to V$

If $\ker T \cap \im T = \{0\}$
then $V = \ker T \oplus \im T$

This is because, consider the subspace $W$ spanned by $\ker T$ and $\im T$. $\ker T$ and $\im T$ become linearly independent subspaces of $W$

$\dim W = \dim \ker T + \dim \im T$
$= \text{nullity } T + \text{rank } T$
$= \dim V$

Therefore $V = W = \ker T \oplus \im T$. 
Suppose \( V = W_1 \oplus W_2 \), and \( W_1 \) and \( W_2 \) are both \( T \)-invariant subspaces. Choose \( \{ d_1, \ldots, d_r \} \) a basis of \( W_1 \), \( \{ \beta_1, \ldots, \beta_s \} \) a basis of \( W_2 \). The matrix of \( T \) relative to \( \{ d_1, \ldots, d_r, \beta_1, \ldots, \beta_s \} \) is

\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\]

\( A \), \( r \times r \) matrix, the matrix for \( T_{W_1} \) relative to \( \{ d_1, \ldots, d_r \} \)

\( B \), \( s \times s \) matrix, the matrix for \( T_{W_2} \) relative to \( \{ \beta_1, \ldots, \beta_s \} \)
This is because for each $\alpha_i$:

\[ T\alpha_i = T_{w_1} \alpha_i = \text{linear combination of } \alpha_i \text{'s.} \]

for each $\beta_i$:

\[ T\beta_i = T_{w_2} \beta_i = \text{linear combination of } \beta_i \text{'s.} \]

Example: $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

to the standard basis defined by $A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$

The minimal polynomial of $A$ is $(x-1)(x-2)^2$

$\Rightarrow A$ is not diagonalizable.
How to simplify $T$?

$W_1 = \ker (I - A) = V_1$

spanned by $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

eigenvector for eigenvalue $1$

$W_2 = \ker (2I - A)^2$

$= \ker \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 0 \end{pmatrix}$

spanned by $\beta_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\beta_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$W_1 \cap W_2 = \{ \mathbf{0} \}$

Since if $\alpha \in W_1 \cap W_2$

$T\alpha = \alpha$
\[ 0 = (2I - A)^2 \alpha = f(1) \alpha \]
\[ f(x) = (2 - x)^2 \]
\[ = (2 - 1)^2 \alpha \]
\[ = \alpha \]
\[ \Rightarrow W_1 \cap W_2 = \{ 0 \} \]
\[ \Rightarrow V = \mathbb{R}^3 = W_1 \oplus W_2 \]

Check both \( W_1 \) and \( W_2 \) are \( T \)-invariant

\( W_1 = \) eigenspace with eigenvalue 1

\[ W_2 = \{ f(T) \beta_2 \}
\]
\[ f(x) \text{ any polynomial} \]

\[ Tw_1 = (1) \text{ relative to } \alpha_1 \]

\[ Tw_2 = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} \text{ relative to } \{ \beta_1, \beta_2 \} \]
Relative to \( \{ \alpha_1, \beta_1, \beta_2 \} \)

\( T \) is represented by

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & -1 \\
0 & 0 & 2
\end{pmatrix}
\]

much simpler!

Or use transformation of basis

\[
P = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
2 & 2 & 1
\end{pmatrix}
\]

\[
P^{-1} A P = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & -1 \\
0 & 0 & 2
\end{pmatrix}
\]

In general \( T: V \rightarrow V \)

\[V = W_1 \oplus \cdots \oplus W_k\]

direct sum of \( T \)-invariant subspaces
The matrix is simple
\[
\begin{pmatrix}
A_1 & 0 \\
A_2 & \ddots & 0 \\
0 & \cdots & A_k
\end{pmatrix}
\]

What are the simplest forms?
The Jordan forms.