Nov 2.

If a linear operator $T$ is diagonalizable, we can find a basis consisting of eigenvectors of $T$.

The matrix of $T$ relative to this basis is
Here we assume $\lambda_i + \lambda_j$ (i ≠ j) and $d_i = \dim V_{\lambda_i}$ - eigenspace.

In this case, it's easy to see

$$\det (xI - T) = (x - \lambda_1)^{d_1} \cdots (x - \lambda_k)^{d_k}$$
In example 2, the characteristic polynomial is \((x-1)(x-2)^2\).

\[ 2 \neq 1 = \text{dim } V_2 = \text{dim } \left( \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) \]

\(A\) is not diagonalizable.

Lemma: Suppose \(T \alpha = \lambda \alpha \) \(\lambda \in \mathbb{F}\).

If \(f\) is any polynomial, then

\[ f(T) \alpha = f(\lambda) \cdot \alpha \]

Proof: It's straightforward to check.

For example, \(f(x) = x^2 + x\)

\[ f(T) \alpha = (T^2 + T) \alpha \]

\[ = T^2 \alpha + T \alpha = T(T \alpha) + \lambda \alpha \]

\[ = T(\lambda \alpha) + \lambda \alpha = \lambda T \alpha + \lambda \alpha \]
\[
= \lambda \alpha + \lambda \alpha = (\lambda^2 + \lambda) \alpha \\
= f(\lambda) \alpha.
\]

Lemma: Fix a linear operator $T$.

If $\alpha_1, \ldots, \alpha_k$ are eigenvectors of $T$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$
then $\{\alpha_1, \ldots, \alpha_k\}$ is linearly independent.

Proof: Assume a linear relation

\[
c_1 \alpha_1 + \cdots + c_k \alpha_k = 0
\]

Then for any polynomial $f(x)$

\[
0 = f(T)(c_1 \alpha_1 + \cdots + c_k \alpha_k)
\]
\[ = c_1 f(T) \alpha_1 + \ldots + c_k f(T) \alpha_k \]
\[ = c_1 f(\lambda_1) \alpha_1 + \ldots + c_k f(\lambda_k) \alpha_k \]

**Fact:** We can choose polynomial \( f_i(x) \) such that
\[ f_i(\lambda_i) = 1 \]
\[ f_i(\lambda_j) = 0 \quad \text{for } j \neq i \]

Then \[ c_i \alpha_i = 0 \]

\[ \Rightarrow c_i = 0 \quad \text{for each } i \]
Theorem: \( T : V \to V \) linear operator. Let \( \lambda_1, \ldots, \lambda_k \) be distinct eigenvalues of \( T \) and \( V_{\lambda_i} \) be corresponding eigenspace \( 1 \leq i \leq k \). The following statements are equivalent:

i) \( T \) is diagonalizable

ii) The characteristic polynomial of \( T \) is
\[
    f = (x - \lambda_1)^{d_1} \cdots (x - \lambda_k)^{d_k}
\]
and \( \dim V_{\lambda_i} = d_i \)

iii) \( \dim V_{\lambda_1} + \cdots + \dim V_{\lambda_k} = \dim V \)
Proof: We have seen $i) \Rightarrow ii)$

(ii) $\Rightarrow$ (iii) follows from the fact that the degree of the characteristic polynomial is $\dim V$

(iii) $\Rightarrow$ i) For each $V_{\lambda_i}$ choose a basis $\{\alpha_i, \ldots, \alpha_{k_i}\}$

Combine all these subsets, we get a subset of $d_1 + \cdots + d_k = \dim V$

elements, denote by $B$.

We claim $B$ is linearly independent.

Consider any linearly relation

It will have the form
\[ \beta_1 + \beta_2 + \cdots + \beta_k = 0, \]
where \( \beta_i \) is a linear combination of \( d_1, \ldots, d_i \), thus in \( V_{i}. \)

By the previous lemma,
\[ \beta_i = 0 \]

Since \( \{d_1, \ldots, d_i\} \) is a basis of \( V_{i} \), all coefficients are zero.

Therefore the linear relation is trivial.

It follows that \( B \) is linearly independent, thus a basis. \( \square \)
Example: \( T : \mathbb{R}^3 \to \mathbb{R}^3 \)

represented relative to the standard basis by

\[
A = \begin{pmatrix}
5 & -6 & -6 \\
-1 & 4 & 2 \\
3 & -6 & -4
\end{pmatrix}
\]

Is \( T \) diagonalizable?

We compute eigenvalues and eigenvectors

\[
\det(xI - A) = \begin{vmatrix}
 x-5 & 6 & 6 \\
 1 & x-4 & -2 \\
-3 & 6 & x+4
\end{vmatrix}
\]

\[
= \begin{vmatrix}
 x-5 & 0 & 6 \\
 1 & x-2 & -2 \\
-3 & 2-x & x+4
\end{vmatrix}
\] (Column operation)
\[ \begin{align*} 
\text{row operation} \quad & \Rightarrow \quad \begin{vmatrix} 
X-5 & 0 & 6 \\
1 & X-2 & -2 \\
-2 & 0 & X+2 
\end{vmatrix} \\
\text{cofactor expansion on column 2} \quad & = \quad (X-2) \left[ (X-5)(X+2) + 12 \right] \\
& = \quad (X-2)(X^2 - 3X + 2) \\
& = \quad (X-1)(X-2)^2 \\
\lambda_1 & = 1 \quad \lambda_2 = 2 \\
\lambda_1 & = 1 \quad \begin{pmatrix} 
-4 & 6 & 6 \\
1 & -3 & -2 \\
-3 & 6 & 5 
\end{pmatrix} \sim \begin{pmatrix} 
0 & -6 & -2 \\
1 & -3 & -2 \\
0 & -3 & -1 
\end{pmatrix} \\
\sim & \begin{pmatrix} 
1 & -3 & -2 \\
0 & -3 & -1 \\
0 & 0 & 0 
\end{pmatrix} \sim \begin{pmatrix} 
1 & 0 & -1 \\
0 & -3 & -1 \\
0 & 0 & 0 
\end{pmatrix} 
\end{align*} \]
\[
\begin{align*}
\text{rank} &= 2 \implies \text{multi} = 1 \\
\dim V_1 &= 1 \\
V_1 \text{ spanned by } &\alpha_1 = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \\
\lambda_2 &= 2 \\
\begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{pmatrix} &\sim &\begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\text{rank} &= 1 \implies \text{multi} = 2 \\
\dim V_2 &= 2 \\
V_2 \text{ spanned by } &\alpha_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\
\alpha_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{align*}
\]

Since \( \dim V_1 + \dim V_2 = 3 = \dim \mathbb{R}^3 \),
\( T \) is diagonalizable.
Let \( P = (x_1 \ x_2 \ x_3) \)

\[
= \begin{pmatrix}
3 & 2 & 0 \\
-1 & 1 & 1 \\
3 & 0 & -1 \\
\end{pmatrix}
\]

\( P^{-1} A P = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{pmatrix} \)