Nov 6.

An application

Suppose a 3 x 3 matrix has the characteristic polynomial

\[ f(x) = (x-1)(x-2)^2 \]

What are the possible minimal polynomial polynomials for A?

What do we know about the minimal polynomial?
1) \( p(x) \mid f(x) \)

2) \( p(x) \) and \( f(x) \) have the same roots.

The possible \( f(x) \) are

\[(x-1)(x-2)\]

or \((x-1)(x-2)^2\)

Both are possible.

If \( A \) is diagonalizable, \( A \) is similar to

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

\[p(x) = (x-1)(x-2)\]
\[(A - I)(A - 2I)\]
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
\[
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

If \( A \) is not diagonalizable
We will see that \( A \) has to be similar to
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}
\]
Jordan canonical form
We check that

\[(A - I)(A - 2I)\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \neq 0
\]

But

\[(A - I)(A - 2I)^2\]

\[
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
The minimal polynomial \( p(x) = (x-1)(x-2)^2 \)

We will learn that all possible Jordan canonical forms for the characteristic polynomial \( (x-1)(x-2)^2 \)

are

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 1 \\
0 & 0 & 2 \\
\end{pmatrix}
\]

and Jordan canonical form
determines a matrix up to similarity
Therefore, in this case, the minimal polynomial classifies the matrices/linear operator up to similarity.

Before I continue, I want to go over the big formula for the determinant function. In particular, what are the permutations of...
\[
\det A = \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)} A_{\sigma(n+1)}
\]

We write out each term for \( n = 3 \) \( |S_3| = 6 \)

\[
6 = (1 \ 2 \ 3) \quad A_{11} A_{22} A_{33} \quad 1
\]

\[
6 = (3 \ 1 \ 2) \quad A_{13} A_{21} A_{32} \quad 2
\]

\[
6 = (2 \ 3 \ 1) \quad A_{12} A_{23} A_{31} \quad 3
\]

\[
6 = (3 \ 2 \ 1) \quad -A_{13} A_{22} A_{31} \quad 4
\]

\[
6 = (2 \ 1 \ 3) \quad -A_{12} A_{21} A_{33} \quad 5
\]

\[
6 = (1 \ 3 \ 2) \quad -A_{11} A_{23} A_{32} \quad 6
\]
Some people may learn the formula this way.

But if \( n = 4, 5, \ldots, 100, \ldots \)

or bigger

you see that you just write down columns in all different orders

\[
\sum_{\sigma \in S_n} \text{sgn} (\sigma) A_{1 \sigma(1)} \ldots A_{i \sigma(i)} \ldots A_{n \sigma(n)}
\]
Come back to minimal polynomials.

If $T$ is diagonalizable.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues.

Choose a basis of eigenvectors \( \{ \alpha_1, \ldots, \alpha_n \} \)

If the eigenvalue of $\alpha_i$ is $\lambda_i$

\[
(T - \lambda_i I) \alpha_i = 0
\]

Therefore

\[
(T - \lambda_1 I) \cdots (T - \lambda_k I) \alpha_i = 0
\]

\[1 \leq i \leq n\]
Since \( \{x_1, \ldots, x_n\} \) is a basis
\[(T - \lambda_1 I) \ldots (T - \lambda_k I) x = 0 \]
for any \( x \in V \).

On the other hand, \( \lambda_1, \ldots, \lambda_k \)
are all roots of characteristic polynomials, thus are roots of
the minimal polynomials
Therefore the minimal polynomials
is \( (x - \lambda_1) \ldots (x - \lambda_k) \)

The converse is also true.
Theorem
Let $V$ be a finite dimensional vector space over a field $F$. and $T: V \rightarrow V$ a linear operator.

Then $T$ is diagonalizable if and only if the minimal polynomial for $T$ has the form

$$p(x) = (x - \lambda_1) \cdots (x - \lambda_k)$$

where $\lambda_1, \ldots, \lambda_k$ are distinct elements in $F$.

Proof: Omit. See the textbook.
Example
Every matrix $A$ such that
\[ A^2 - 3A + 2I = 0 \]
is diagonalizable.

Proof:
\[ (A - 2I)(A - I) = 0 \]
The possible minimal polynomials are
\[ x - 2, \]
\[ x - 1, \]
\[ (x - 2)(x - 1) \].
All have the form
\[ (x - \lambda_1) \cdots (x - \lambda_k) \]
for distinct $\lambda_1, \ldots, \lambda_k$. 
Apply the theorem, $A$ is always diagonalizable.

A similar theorem.
Let $V$ be a finite-dimensional vector space over a field $F$, and $T: V \to V$ a linear operator. Then $T$ is triangulable (meaning can be represented by an upper-triangular matrix relative to some basis) if and only if the minimal polynomial for $T$ has the form $(x - \lambda_1)^{r_1} \cdots (x - \lambda_k)^{r_k}$.
For example over $\mathbb{C}$, the field of complex numbers, every square matrix is similar to an upper-triangular matrix.

Not true for $F = \mathbb{R}$.

\[
\begin{pmatrix}
2 & 3 \\
-1 & 1
\end{pmatrix}
\] is not triangularizable over $\mathbb{R}$, but is diagonalizable over $\mathbb{C}$. 