Oct 2

3.1 Linear transformations.
We've studied vector spaces. Now we study the maps between vector spaces.

Definition: Let $V$ and $W$ be vector spaces over the same field $F$. A linear transformation or a linear map from $V$ to $W$ is a function $T : V \rightarrow W$ such that...
for all $\alpha, \beta$ in $V$, and all scalar $c$ in $F$.

\[
T(\alpha + \beta) = T(\alpha) + T(\beta)
\]

\[
T(c\alpha) = cT(\alpha)
\]

Example: $V$ = the space of polynomials over the field $C$

$f \in V$ $f = c_0 + c_1x + \cdots + c_n x^n$  
$c_i \in C$ $0 \leq i \leq n$.

$D : V \rightarrow V$

is taking the derivative

$Df(x) = f'(x)$
Since derivative is linear.
\[ D(f+g) = (f+g)' \]
\[ = f' + g' \]
\[ = Df + Dg \]
\[ D(cf) = (cf)' = cf' \]
\[ = c \cdot Df \]

\( D \) is a linear map.

Example: \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)
\[ T((x_1, x_2)) = (x_1 + x_2, 3x_1) \]
\( \alpha = (a_1, a_2) \)
\( \beta = (b_1, b_2) \) are two vectors in \( \mathbb{R}^2 \)

First compute

\[
T(\alpha + \beta) = T((a_1 + b_1, a_2 + b_2))
\]

\[= (a_1 + b_1 + a_2 + b_2, 3a_1 + 3b_1)\]

Compute

\[
T(\alpha) + T(\beta) = T((a_1, a_2)) + T((b_1, b_2))
\]

\[= (a_1 + a_2, 3a_1) + (b_1 + b_2, 3b_1)\]

\[= (a_1 + a_2 + b_1 + b_2, 3a_1 + 3b_1)\]

Therefore, \( T(\alpha + \beta) = T(\alpha) + T(\beta) \)
For $c$ a scalar in $F$.

\[
T(c\alpha) = T((ca_1, ca_2)) \\
= (ca_1 + ca_2, 3ca_1)
\]

\[
c T(\alpha) = c \cdot T((a_1, a_2)) \\
= c (a_1 + a_2, 3a_1) \\
= (ca_1 + ca_2, 3ca_1)
\]

Therefore, $T(c\alpha) = c T(\alpha)$

$T$ is a linear transformation.

Here $x_1 + x_2$, $3x$, are both linear functions, you can not include constants, quadratics or other higher degree terms.
For a linear transformation

\[ T : V \to W \]

\[ T(0) = 0. \]

\( T \) maps a linear combination to a linear combination.

\[ T \left( \sum_{i=1}^{n} c_i \alpha_i \right) = \sum_{i=1}^{n} c_i T(\alpha_i) \]

In particular, if \( \{ \alpha_1, \ldots, \alpha_n \} \) is a basis of \( V \), each vector \( \alpha \in V \) is a linear combination of \( \{ \alpha_1, \ldots, \alpha_n \} \) in a unique way, then the linear combination \( T \) is determined by its values
on \( \alpha_1, \ldots, \alpha_n \). That is
\[
\{ T(\alpha_1), \ T(\alpha_2), \ldots, \ T(\alpha_n) \}
\]

In other words,

Then \( V \) and \( W \) are vector spaces over a field \( F \).
\( \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \) is a basis of \( V \).

Take arbitrary vectors
\( \{ \beta_1, \ldots, \beta_n \} \) in \( W \).

Then there is a \underline{unique} linear transformation \( T : V \rightarrow W \)
such that \( T(\alpha_i) = \beta_i \) \( 1 \leq i \leq n \).

Proof. For any \( \alpha = \sum_{i=1}^{n} c_i \alpha_i \)
Define \( T(\alpha) = \sum_{i=1}^{n} c_i \beta_i \)
It is straightforward to check that $T$ thus defined is a linear transformation. □

For $T: V \rightarrow W$,

There are two subspaces associated with $T$.

One is the image (range), a subspace of $W$.

The other is the kernel, a subspace of $V$. 
Definition:
For a linear transformation $T: V \rightarrow W$.
The image, denoted by $\text{im} T$, is the subset of $W$ of images of the map $T$.

$\text{im} T = \{ \beta \in W \text{ such that } \beta = T(\alpha) \text{ for some } \alpha \in V \}$

The kernel, denoted by $\ker T$, is the subset of $V$ of elements mapped to $0$ by $T$.

$\ker T = \{ \alpha \in V : T(\alpha) = 0 \}$

This means "such that"
The important fact is that both subsets are actually subspaces (Check the conditions)

\[
\text{rank of } T = \text{the dimension of } \text{im } T
\]

\[
\text{nullity of } T = \text{the dimension of } \text{ker } T.
\]

**Important Theorem:**

\[ T: V \rightarrow W \text{ is a linear transformation between vector spaces } V \text{ and } W. \text{ Suppose } V \text{ is finite dimensional.} \]

\[ \text{rank}(T) + \text{nullity}(T) = \dim V \]
Proof: Choose a basis \( \{ d_1, d_2, \ldots, d_k \} \) of \( \ker T = \mathbb{V} \).
Add vector \( d_{k+1}, \ldots, d_n \) to extend the set to a basis \( \{ d_1, d_2, \ldots, d_k, d_{k+1}, \ldots, d_n \} \) of \( \mathbb{V} \). We need to show that \( \{ T(d_{k+1}), \ldots, T(d_n) \} \) is a basis for \( \text{im} T \).
First, \( \{ T(d_{k+1}), \ldots, T(d_n) \} \) spans \( \text{im} T \). This is because \( \{ T(d_1), \ldots, T(d_k), T(d_{k+1}), \ldots, T(d_n) \} \) spans \( \text{im} T \). (The image of basis
determining the image of any vector). And \( T(\alpha_1) \ldots T(\alpha_k) \) are all zero.

Secondly, \( \{T(\alpha_{k+1}), \ldots, T(\alpha_n)\} \) is linearly independent.

Consider any linear relation of \( T(\alpha_{k+1}), \ldots, T(\alpha_n) \)

\[
\sum_{i=k+1}^{n} C_i T(\alpha_i) = 0
\]

\[
T\left(\sum_{i=k+1}^{n} C_i \alpha_i\right) = 0
\]

Therefore \( \sum_{i=k+1}^{n} C_i \alpha_i \) is in \( \ker T \).
That means 
\[ \sum_{i=k+1}^{n} Ci \, di \] is in the span of \( d_1, d_2, \ldots, d_k \).

But \( \{ d_1, \ldots, d_k, d_{k+1}, \ldots, d_n \} \)

is linearly independent.

Therefore \( C_{k+1} = \cdots = C_n = 0 \)

\( \{ T(d_{k+1}), \ldots, T(d_n) \} \)

is linearly independent.

\[ \text{rank}(T) = n - k \]

\[ \text{nullity}(T) = k \]

\( \dim V = n \) \( \square \)
rank of a matrix $A$. An $m \times n$ matrix $A$ defines a linear transformation

$$T : F^n \rightarrow F^m$$

If $\alpha = (a_1, \ldots, a_n) \in F^n$,

Define $T(\alpha) = A \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

a vector in $F^m$.

Definition. $\text{rank } A = \text{rank } T$.

**Theorem:**

$\text{rank } A = \dim \text{ of row space of } A$

$= \dim \text{ of column space of } A$.
By definition.
The column space of $A$

$= \text{im } T.$

Therefore

$\text{rank } A = \text{rank } T$

$= \text{dim of the column space}$

On the other hand

$\ker T = \text{the space of solutions of } A \tilde{x} = 0$

What is the dimension of the space of solutions of $A \tilde{x} = 0$?
Consider $R$, the row-reduced echelon matrix that is row-equivalent to $A$.

Assume $R$ has $r$ nonzero rows, the pivots are at positions $k_1 < k_2 < \ldots < k_r$.

Consider solutions for $R\tilde{x} = 0$.

$x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ are expressed in terms of other $n-r$ variables.

These are called free variables.

These $n-r$ variables can take
arbitrary values in $F$. Therefore the dim of the space of solutions is $n - r$.

\[ \text{nullity } T = n - r \]

Since

\[ \text{rank } T + \text{nullity } T = \dim V = n \]

\[ \text{rank } A = \text{rank } T = r \]

However $r = \dim$ of the row space of $A$

We have $\text{rank } A = \dim$ of the row space of $A$
\[F^n \xrightarrow{T} F^m\]

\[A\]

- The row space of \(A\)
- \(\text{ker} T = \text{solutions of } A\mathbf{x} = 0\)
- \(n - r\) solutions of \(A\mathbf{x} = 0\)
- \(r = \text{rank } A = \text{rank } T\)
- Or we have \(A^T\mathbf{x} = 0\)

- The column space of \(A\)
- \(\text{im } T = \text{solutions of } \mathbf{y}A = 0\)
- \(m - r\) solutions of \(\mathbf{y}A = 0\)