Another formula: cofactor formula.

Definition: $n > 1$, $A$ an $n \times n$ matrix. We let $A(i \mid j)$ denote the $(n-1) \times (n-1)$ matrix obtained by deleting the $i$th row and $j$th column of $A$. If $D$ is a multilinear
function on the set of \((n-1) \times (n-1)\) matrices, we put

\[
D_{ij}(A) = D[A(i_{ij})]
\]

Theorem

Let \(n > 1\), and \(D\) a multilinear function on \((n-1) \times (n-1)\) matrices. For each \(j\), \(1 \leq j \leq n\) define the function on \(n \times n\) matrices

\[
E_{j}(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)
\]

If \(D\) is the determinant, then \(E_{j}\) is the determinant.
Proof: Again it suffices to check the three properties.

i) \( A = I_n \)

then \( I_n | (ij|j) = I_{n-1} \)

If \( i \neq j \) off diagonal
\[ A_{ij} = 0 \]

Therefore \( \text{E}_{ij}(I_n) = (-1)^{j+1} \times | \]
\[ \times \det(I_{n-1}) \]
\[ = 1 \]

ii) \( D_{ij} \) is independent of the \( i \)th row of \( A \) and is linear for the rest \((n-1)\) rows
$A_{ij} D_{ij}(A)$ is then also linear for row $i$.

Therefore $(-1)^{i+j} A_{ij} D_{ij}(A)$ is multilinear.

Take the sum, $E_j$ is still multilinear.

iii) Suppose row $r = \text{rows } r < s$.

If $i \neq r$, $i \neq s$, then $A_{i(rj)}$ has two rows equal, and $D_{ij}(A) = 0$.

Therefore

$$E_j(A) = (-1)^{r+j} A_{rj} D_{rj}(A) + (-1)^{s+j} A_{sj} D_{sj}(A).$$
Suppose $a$ is or deleting $A_{rj}$

$$A(rlj) \text{ and } A(slj) \text{ are almost the same expect}
A(rlj) \text{ has } a \text{ in row } (s-1)
A(slj) \text{ has } a \text{ in row } r$$

In order to get $A(rlj)$ from $A(slj)$ we need to exchange the rows $(s-1-r)$ times

$$\det(A(rlj)) = (-1)^{s-1-r} \det(A(slj))$$
\[ E_j (A) = (-1)^{r+j} (-1)^{s-1-r} A_{sj} D_{sj} (A) + (-1)^{s+j} A_{sj} D_{sj} (A) = 0 \]

Therefore \( E_j = \det A \)

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Example \( n=3 \).

Use cofactor formula for \( j=2 \).

\[ A = \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{pmatrix} \]

\[ \det A = E_2 (A) = (-1)^3 A_{12} \]

\[ \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + (-1)^4 A_{21} \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} \]
\[\begin{align*}
&\quad + (-1)^5 A_{32} \begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \\
&= \quad - A_{11} (A_{21} A_{33} - A_{31} A_{23}) \\
&\quad + A_{22} (A_{11} A_{33} - A_{31} A_{13}) \\
&\quad - A_{32} (A_{11} A_{23} - A_{21} A_{13}) \\
&= \quad A_{11} A_{22} A_{33} + A_{12} A_{23} A_{31} \\
&\quad + A_{13} A_{21} A_{32} - A_{12} A_{21} A_{33} \\
&\quad - A_{13} A_{22} A_{31} - A_{11} A_{23} A_{32} \\
&= \sum_{6 \in S_3} \text{sgn}(6) A_{16(1)} A_{26(2)} A_{36(3)}
\end{align*}\]
\((-1)^{i+j} \det A(i\mid j)\) is called the \(i,j\) cofactor of \(A\).

\[
\det A = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det A(i\mid j)
\]

is called the expansion of \(\det A\) by cofactors of the \(j\)th column.