Oct 28

Computing determinants

Three ways

1. \[ \det A = \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)} \ldots A_{\sigma(n)} \]

2. Cofactor formula

3. Simplify the matrix by row operations/column operations

   \[ \Rightarrow \text{ upper triangular} \]

   \[ \rightarrow \text{ lower triangular matrices} \]

Use 1)
\[ \det \begin{pmatrix} a_1 & \star & \cdots & \star \\ \vdots & & \ddots & \vdots \\ 0 & & & a_n \end{pmatrix} = a_1 \cdots a_n \]

\[ = \det \begin{pmatrix} a_1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{pmatrix} \]

Example:

\[ \det \begin{pmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{pmatrix} \]

= operation ii) \[ \det \begin{pmatrix} 1-t^2 & 0 & 0 \\ t & 1-t^2 & t \end{pmatrix} \]

= \[ \det \begin{pmatrix} 1-t^2 & t & 0 \\ 0 & 1-t^2 & 0 \\ 0 & 0 & t \end{pmatrix} \]
\[ \text{cofactor} \quad (1 - t^2) \det \begin{pmatrix} 1 & 0 \\ t & 1 + t^2 \end{pmatrix} \]

\[ = (1 - t^2)(1 - t^2) \]

\underline{Conclusion}

Since \( \det A = \det A^t \)

You can also use column operation and expansion of cofactor of rows.
An \( nxn \) matrix \( A \) over \( \mathbb{C} \), the field of complex numbers, is said to be unitary if

\[
A \cdot A^* = I \quad (A^* = A^t)
\]

2f \( A \) is unitary, show that

\[
|\det A| = 1
\]

Proof.

Since \( \det A \)

\[
= \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{1,\sigma(1)} \cdots A_{n,\sigma(n)}
\]

\[
= \frac{1}{\det A} \left( \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{(1),\sigma(1)} \cdots A_{n,\sigma(n)} \right)
\]

\[
= \frac{1}{\det A}
\]
det \ A^* = \ det \ \bar{A}^t \\
= \ det \ \bar{A} \\
= \ \frac{\det \bar{A}}{\det A}

\text{Since} \quad AA^* = 1

1 = \ det \ I = \ det \ AA^* \\
= \ det A \ det A^* \\
= \ (\det A) (\det A) \\
= \ |\det A|^2

\text{Since} \quad |\det A| \geq 0

|\det A| = 1
Example: Let n be an even number. Consider an \( n \times n \) matrix \( A \), such that all entries on the diagonal are even integers, all off diagonal are odd integers. Show that \( A \) is invertible as a matrix over \( \mathbb{R} \).

Proof: We want to show that \( \det A \) is an odd number, thus not zero.

Consider
\[ \det A = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(n)} \]

We know that if 
\( \sigma \) fixes some element 
\( i \in \{1, \ldots, n\} \)

\( \sigma(i) = i \)

then \( A_{\sigma(i)} \) is even

and \( \mathrm{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(n)} \)

is even.

Otherwise if \( \sigma \) does not fix any element, all \( A_{\sigma(i)} \) are off diagonal

\( \mathrm{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(n)} \) is odd.
If a permutation does not fix any element, i.e.,

\[ \sigma(i) \neq i \text{ for any } i \]

it is called a derangement.

Example

\[ \begin{align*}
\text{fix} & \ 1 \ 2 \ 3. \\
\sigma & = (1 \ 2) \\
\sigma & = (2 \ 1)
\end{align*} \]

\[ \begin{align*}
\text{fix} & \ 1 \ 2 \ 3. \\
\sigma & = (1 \ 3 \ 2) \\
\sigma & = (2 \ 1 \ 3) \\
\sigma & = (3 \ 2 \ 1)
\end{align*} \]

\[ D_n \ (\text{or } !n) = \# \text{ of derangements in } S_n. \]

How to compute \( !n \)?
$D_2 = 1 \iff \text{odd}$

$D_3 = 2 \iff \text{even}$

**Lemma** For $n \geq 3$

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$$

**Proof:** Let $6$ be a derangement

$$6(1) = i \neq 1$$

So $i$ could be $2, \ldots, n$

$(n-1)$ choices

For remaining numbers, rearrange them

$$i, 2, \ldots, (i-1), i+1, \ldots, n$$

$6$

$$i, 2, \ldots, (i-1), i+1, \ldots, n.$$
There are two cases

Either \( \delta(i) \neq 1 \)

\[
\begin{align*}
  i, & \quad 2, & \quad \ldots & \quad n \\
  \downarrow & & & \downarrow \\
  1, & \quad 2, & \quad \ldots & \quad n
\end{align*}
\]

\( \delta \) is not allowed

There are \( D_{n-1} \) of these.

Or \( \delta(i) = 1 \)

\[
\begin{align*}
  i, & \quad 2, & \quad \ldots & \quad n \\
  \downarrow & & & \downarrow \\
  1, & \quad 2, & \quad \ldots & \quad n
\end{align*}
\]

There are \( D_{n-2} \) of these

\[ D_n = (n-1) (D_{n-1} + D_{n-2}) \]
Lemma

\( D_n \) is even if \( n \) odd

\( D_n \) is odd if \( n \) even

Proof

\( n = 2 \) \( \Rightarrow D_n \) odd

\( n = 3 \) \( \Rightarrow D_n \) even

Assume \( n = 2k \) \( \Rightarrow D_{2k} \) odd

\( n = 2k+1 \) \( \Rightarrow D_{2k+1} \) even

Then \( D_{2k+2} = (2k+1)(D_{2k} + D_{2k+1}) \)

\( \uparrow \)

odd \quad odd

is odd

\( D_{2k+3} = (2k+2)(D_{2k+2} + D_{2k+1}) \)

\( \uparrow \)

even

is even
By induction, the lemma is proved.

Now if \( n \) is even, \( D_n \) is odd

\[
\det A = \sum_{6 \in S_n} \text{sgn}(6) A_{6(1)} \ldots A_{6(n)}
\]

\[
= \sum_{\text{even}} + \sum_{\text{odd}}
\]

Add odd terms of odd numbers

\[
\Rightarrow \text{odd}
\]

Therefore \( \det A \) is odd \( \neq 0 \)

It follows that \( A \) is invertible because we are over \( \mathbb{R} \).