Bases and dimension

Definition:
A linear relation among vectors $a_1, a_2, \ldots, a_n$ is any linear combination that evaluates to zero: any equation of the form

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$$

If $a_i$ are column vectors

$$
\begin{pmatrix}
  a_{1i} \\
  a_{2i} \\
  \vdots \\
  a_{mi}
\end{pmatrix}
$$
\[ x_1 \left( \begin{array}{c} a_{11} \\ \vdots \\ a_{1n} \end{array} \right) + x_2 \left( \begin{array}{c} a_{21} \\ \vdots \\ a_{2n} \end{array} \right) + \cdots + x_n \left( \begin{array}{c} a_{1n} \\ \vdots \\ a_{mn} \end{array} \right) = 0 \]

means \[ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \] is a solution for \( A \vec{x} = 0 \).
Therefore a linear relation for $\alpha_1, \ldots, \alpha_n$ is also written as

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)X = 0$$

for $X$ a column vector in $F^n$.

**Definition:** A set of vectors $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is called linearly independent if any linear relation
\[ x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0 \]

implies that \( x_1 = x_2 = \cdots = x_n = 0 \)

In other words, the equation

\[
( a_1 a_2 \cdots a_n ) X = 0
\]

has only trivial solution.

A set that is not linearly independent is called linearly dependent. In other words,

\[
( a_1 a_2 \cdots a_n ) X = 0
\]
has nontrivial solutions.

Example: \( V = \mathbb{R}^3 \), \( F = \mathbb{R} \)

\[
\begin{align*}
\alpha_1 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
\alpha_2 &= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\
\alpha_3 &= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \\
\alpha_4 &= \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}
\end{align*}
\]

\( \{ \alpha_1, \alpha_2, \alpha_3 \} \) is linearly independent.

\( \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) is linearly dependent.
\[
\begin{pmatrix}
1 & 1 & 2 \\
0 & 2 & 1 \\
1 & 0 & 2
\end{pmatrix}
\text{ is invertible,}
\]

\[
\begin{pmatrix}
1 & 1 & 2 & 1 \\
0 & 2 & 1 & 1 \\
1 & 0 & 2 & 3
\end{pmatrix}
\text{ for } m < n
\]

3 equations 4 unknowns 
there is always a non-trivial solution.

Geometrically \( x_1, x_2, x_3 \) are 
not in the same plane.
Definition. V a vector space over F. A basis for V is an ordered set, that is linearly independent and also spans V. A vector space is finite dimensional if it has a finite basis.

Example: \( F^n \) has a standard basis

\[ S = \{ e_1, e_2, \ldots, e_n \} \]

\[ e_i = (0, \ldots, 1, 0, \ldots, 0) \]

\[ \text{↑} \]

\[ i \]
\[ 0 = x_1e_1 + x_2e_2 + \cdots + x_ne_n \]
\[ = (x_1, x_2, \ldots, x_n) \]

implies that \( x_1 = x_2 = \cdots = x_n = 0 \)

So \( S \) is linearly independent.

Any \( \alpha = (x_1, x_2, \ldots, x_n) \in \mathbb{F}^n \)

\[ = x_1e_1 + x_2e_2 + \cdots + x_ne_n \]

So \( S \) spans \( \mathbb{F}^n \)

Example Let \( V = \mathbb{R}^n \).

\( \alpha_1 \ldots \alpha_n \) are column vectors of an \( n \times n \) matrix \( A \).

\( \{\alpha_1, \ldots, \alpha_n\} \) is a basis if and
only if \( A \) is invertible

Proof: \( A \) is invertible

\[
\begin{align*}
\Rightarrow & \quad A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution} \\
\Leftarrow & \quad A\mathbf{x} = \mathbf{y} \text{ has a solution for every } \mathbf{y} \in \mathbb{F}^n \\
\end{align*}
\]

\[
\begin{align*}
\Rightarrow a) & \quad \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \text{ is linearly independent} \\
\Leftarrow b) & \quad \text{Any } \mathbf{y} \in \mathbb{F}^n \text{ is in the span of } S. \\
\end{align*}
\]