1 Determinants

1. (Cauchy Determinant) Let $A = [a_{ij}]$ be an $n \times n$ matrix where
\[
a_{ij} = \frac{1}{x_i + y_j}, \quad 1 \leq i \leq n, 1 \leq j \leq n.
\]
Show that
\[
\det A = \frac{\prod_{i>j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(x_i - y_j)}
\]

2. What is the determinant of the following $n \times n$ matrix?
\[
\begin{bmatrix}
1 & 0 & 1 & \ldots & 1 \\
\lambda & 1 & \lambda_1 & \ldots & \lambda_{n-2} \\
\lambda^2 & 2\lambda & \lambda_1^2 & \ldots & \lambda_{n-2}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda^{n-1} & (n-1)\lambda^{n-2} & \lambda_1^{n-1} & \ldots & \lambda_{n-2}^{n-1}
\end{bmatrix}
\]

Hint: This is an example of a confluent Vandermonde determinant; try a few small cases may enable you to intuit the answer. However to get the full solution, remember that a determinant of a matrix is a linear function of its columns. Therefore, if exactly one column of a matrix has entries which are functions of some variable, then the determinant of the matrix obtained by taking the derivatives of the elements in that column is the same as the derivative of the determinant. This is related to the problem of linearly recurrent sequences. Recall that such a sequence with recurrence length $n$ is a sequence of numbers $f_0, f_1, f_2, \ldots$ such that
\[
f_i = a_1f_{i-1} + a_2f_{i-2} + \ldots + a_nf_{i-n}
\]
for some fixed numbers $a_1, a_2, \ldots, a_n$. The whole infinite sequence is then determined by its starting segment $f_0, f_1, \ldots, f_n$ which must be given in advance.
2 Linear recurrence

3. a) Give three linearly independent recursive sequences satisfying the relation
\[ f_{n+1} = 4f_n - 5f_{n-1} + 2f_{n-2} \]

b) Suppose, more generally that
\[ f_i = a_1 f_{i-1} + a_2 f_{i-2} + a_3 f_{i-3} \]
and that the characteristic polynomial of the recursion matrix is \((x - \lambda)^2(x - \lambda_1)\). Give three linearly independent recursive sequences satisfying the relation.

Hint: See the previous problem with \(n = 3\).

3 Tridiagonal Matrices

A tridiagonal matrix is a square matrix \( A = [a_{ij}] \), where \( a_{ij} = 0 \) for \(|i - j| > 1\). Let \( a_i = a_{ii} \) for \( i = 1, \ldots, n \), let \( b_i = a_{i,i+1} \) and \( c_i = a_{i+1,i} \) for \( i = 1, \ldots, n - 1 \). Then the tridiagonal matrix takes the form

\[
\begin{bmatrix}
a_1 & b_1 & 0 & \ldots & 0 & 0 & 0 \\
c_1 & a_2 & b_2 & \ldots & 0 & 0 & 0 \\
0 & c_2 & a_3 & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & a_{n-2} & b_{n-2} & 0 \\
0 & 0 & 0 & \ldots & c_{n-2} & a_{n-1} & b_{n-1} \\
0 & 0 & 0 & \ldots & 0 & c_{n-1} & a_n
\end{bmatrix}
\]

4. Let \( \theta = \pi/4 \) and set
\[
S = \frac{1}{\sqrt{2}} \begin{bmatrix}
\sin \theta & \sin 2\theta & \sin 3\theta \\
\sin 2\theta & \sin 4\theta & \sin 6\theta \\
\sin 3\theta & \sin 6\theta & \sin 9\theta
\end{bmatrix}
\]

Show that \( S \) is orthogonal and that its columns are the eigenvectors of any matrix of the form
\[
A = \begin{bmatrix}
a & b & 0 \\
b & a & b \\
0 & b & a
\end{bmatrix}
\]

The matrix \( A \) is a tridiagonal Toeplitz matrix. Show that the eigenvalues of \( A \) are \( a + 2b \cos k\theta \), \( k = 1, 2, 3 \). What is the generalization to tridiagonal Toeplitz matrices of arbitrary size?
5. Now let

\[ B = \begin{bmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{bmatrix}, \]

where \( b \) and \( c \) both have the same sign (i.e., both positive or both negative). Show that there is a diagonal matrix \( D \) such that \( B = DAD^{-1} \). Use this to find the eigenvalues and eigenvectors of \( B \), and give the generalization to tridiagonal Toeplitz matrices of arbitrary size.

4 Applications to Continued Fractions

6. Let’s explore the relation between determinants and continued fractions, combining the determinants, linear recurrence and tridiagonal matrices. Recall a tridiagonal matrix takes the form

\[
\begin{bmatrix}
  a_1 & b_1 & 0 & \ldots & 0 & 0 & 0 \\
  c_1 & a_2 & b_2 & \ldots & 0 & 0 & 0 \\
  0 & c_2 & a_3 & \cdot & \cdot & \cdot & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \cdot & a_{n-2} & b_{n-2} & 0 \\
  0 & 0 & 0 & \ldots & c_{n-2} & a_{n-1} & b_{n-1} \\
  0 & 0 & 0 & \ldots & 0 & c_{n-1} & a_n \\
\end{bmatrix}
\]

Restrict the matrix \( A \) to its first \( k \) rows and first \( k \) columns is still a \( k \times k \) tridiagonal matrix. Let’s denote its determinant by \( \Delta_k \), and \( \Delta_0 = 1 \).

a) Verify the recurrence relation

\[ \Delta_k = a_k \Delta_{k-1} - b_{k-1} c_{k-1} \Delta_{k-2} \]

for \( k > 2 \).

b) Focus on a special case \( c_i = -1, b_i = 1 \) for \( i = 1, \ldots, n - 1 \). Denote the determinant

\[
(a_1 \ldots a_n) = \begin{vmatrix}
  a_1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
  -1 & a_2 & 1 & \ldots & 0 & 0 & 0 \\
  0 & -1 & a_3 & \ddots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \ddots & a_{n-2} & 1 & 0 \\
  0 & 0 & 0 & \ldots & -1 & a_{n-1} & 1 \\
  0 & 0 & 0 & \ldots & 0 & -1 & a_n
\end{vmatrix}
\]
Show that \( (a_1a_2\ldots a_n)/(a_2a_3\ldots a_n) \) is equal to the following expression

\[
a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}. \tag{1}
\]

c) [This is a remark. ] If \( a_1 \) is an integer, and \( a_i \) a positive integer for \( i = 2, \ldots, n \), this number (1) is called a finite continued fraction and is denoted by \([a_1; a_2, a_3, \ldots, a_n]\). Each rational number \( \frac{p}{q} \) can be expressed as a finite continued fraction like this. On the other hand, each irrational number can be expressed as an infinite continued fraction like

\[
a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}. \tag{2}
\]

For example the infinite continued fraction expression for \( \pi \) is

\([3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots]\).

The terms in this expression are apparently random. So this gives you a way of generating random numbers. What we learnt here is we can associate any real number a tridiagonal matrix over integers. For rational numbers, this is a finite matrix, while for an irrational number this is an infinite matrix.

d) Now further if \( a_i = 1 \) for each \( i = 1, \ldots, n \), we get \( \Delta_k = \Delta_{k-1} + \Delta_{k-2} \). So these are Fibonacci numbers. Assume the fact that the golden ratio \( \phi \) is the limit

\[\phi = \lim_{k\to\infty} \frac{\Delta_k}{\Delta_{k-1}}.\]

Write the golden ratio \( \phi \) as an infinite continued fraction.

### 5 Applications in Analysis

7. Let \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) be vectors in a vector space (possibly infinite dimensional) with an inner product \( \langle \cdot, \cdot \rangle \). Define the Gram determinant by

\[G(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det(\langle \mathbf{v}_i, \mathbf{v}_j \rangle)\]
a) If the \( v_1, \ldots, v_k \) are orthogonal, compute their Gram determinant.

b) Show that the \( v_1, \ldots, v_k \) are linearly independent if and only if their Gram determinant is not zero. (Hint: I actually did this when I was inducing the projection formula.)

c) Better yet, if the \( v_1, \ldots, v_k \) are linearly independent, show the the symmetric matrix \( (\langle v_i, v_j \rangle) \) is positive definite. In particular, \( k = 2 \), \( G(v_1, v_2) \geq 0 \) is the Cauchy-Schwarz inequality.

d) If \( v_1, \ldots, v_k \) are orthogonal vectors obtained from linear independent vectors \( w_1, \ldots, w_k \) by Gram-Schmidt. What is the relationship between \( G(v_1, \ldots, v_k) \) and \( G(w_1, \ldots, w_k) \)?

e) Let \( S \) denote the subspace spanned by the linearly independent vectors \( w_1, \ldots, w_k \). If \( \vec{\alpha} \) is any vector, let \( \text{Proj}_S \vec{\alpha} \) be the orthogonal projection of \( \vec{\alpha} \) onto \( S \), prove that the distance \( ||\vec{\alpha} - \text{Proj}_S \vec{\alpha}|| \) from \( \vec{\alpha} \) to \( S \) is given by the formula

\[
||\vec{\alpha} - \text{Proj}_S \vec{\alpha}||^2 = \frac{G(\vec{\alpha}, w_1, \ldots, w_k)}{G(w_1, \ldots, w_k)}. \tag{3}
\]

8. (Continue) We have actually seen the above formula in Math 114.

a) Show that in \( \mathbb{R}^2 \), the formula for the distance from a point to a line is a special case of the above formula (3).

b) Show that in \( \mathbb{R}^3 \), the formula for the distance from a point to a plane is a special case of the above formula (3).

9. (Continue) We now apply the formula 3 to prove Muntz’s theorem.

We also need the Cauchy determinants. Let \( A \) be an \( n \times n \) square matrix with \( A_{ij} = \frac{1}{a_i + b_j}, 1 \leq i, j \leq n \). Then we have

\[
\det A = C(a_1, \ldots, a_n, b_1, \ldots, b_n) = \frac{\prod_{i=1}^n \prod_{j=1}^{i-1} (a_i - a_j)(b_i - b_j)}{\prod_{i=1}^n \prod_{j=1}^i (a_i + b_j)}
\]

The Question: Consider the space of continuous real functions on \([0, 1]\), with the inner product \( \langle f, g \rangle := \int_0^1 f(x)g(x)dx \) and related norm \( ||f||^2 = \langle f, f \rangle \). Let \( \mathcal{S}_k \) be the subspace spanned by functions \( \{x^{n_1}, \ldots, x^{n_k}\} \), where \( \{n_1, \ldots, n_k\} \) are distinct positive integers. Let \( h(x) := x^l \) where \( l > 0 \) is a positive integer— but not one of the \( n_j \)'s. Prove that

\[
\lim_{k \to \infty} ||h - \text{Proj}_{\mathcal{S}_k} h|| = 0
\]

if and only if \( \sum \frac{1}{n_j} \) diverges. (Hint: Here you need to use the fact that \( \prod_j (1 - a_j) \) goes to 0, if and only if \( \sum_j a_j \) diverges.) This, combined with the Weierstrass Approximation theorem, proves Muntz’s Theorem: Linear combinations of \( x^{n_1}, \ldots, x^{n_k}, \ldots \) are dense in \( L^2(0, 1) \) if and only if \( \sum \frac{1}{n_j} \) diverges.
6 Applications in Random Walk

10. Suppose that a particle moves on the real line starting at time $t = 0$ at one of the integer points 1, 2, 3, or 4. After one second it moves one unit right or left, each with probability $1/2$. If it reaches either the point 0 or the point 5 it is absorbed and ceases to move. Ultimately it will be absorbed either at 0 or at 5. Determine the probability of it being absorbed at 0 as a function of the point at which it started, as well as the expected average time until it is absorbed (This is called the expectation of the stop time). This is a standard problem, which you should be able to find on the web. Just google gambler’s ruin or matching pennies.

The problem can be solved in many ways; one is by setting up a simple recursion which can be found in many places, the other by examining the Markov matrix which controls the random process involved. A central point you should discover if you follow all the problems is that any linear recurrence relation could be expressed by matrices. Of course the most elegant solution is provided by the idea of Martingales, but that is another topic.

For full credit, use the (admittedly more complicated but generalizable) Markov method. The computations in this small case can be done by hand.

11. Now solve the preceding problem with 5 replaced by an arbitrary $N \geq 5$. This is equivalent to the game of matching pennies when the total number of pennies in the game (the preceding $m + n$) is $N$.

12. Finally, solve the problem with general $N$, but where the particle, if not already absorbed, has a probability $p$ of moving one unit to the right and a probability $q = 1 - p$ of moving one unit to the left. You may find the formula online, but the point is how to calculate it using the Markov matrix.

An important case for applications is that where there is no absorption on the right ($N \to \infty$); the particle can travel arbitrarily far but still randomly moves back and forth. Is it still certain to be absorbed or is there some positive probability that it never will be? (Think of matching pennies against an opponent with infinitely many pennies; it is certain to be absorbed if $p \leq 1/2$.) On the average, what is the expected average time that the particle “lives”, i.e., until it is absorbed, if it is. (The value may seem surprising when $p = 1/2$.)

7 Adjoint and Inverses

13. Assume $A$ and $B$ are invertible. Show the following statements are true
a) $\text{adj}(AB) = \text{adj} B \cdot \text{adj} A$; 

b) $\text{adj} XAX^{-1} = X(\text{adj} A)X^{-1}$; 

c) if $AB = BA$, then $(\text{adj} A)B = B(\text{adj} A)$.

(Hint: these properties are very close to the properties of inverses.)

Since the above statements are basically polynomial equations for entries of $A$ and $B$, once we show they are true for all invertible matrices $A$ and $B$, we can show that they are true for general $A$ and $B$ by a standard argument (called deformation to the invertible matrices). You don’t have to make this argument. Just accept the following theorem as a corollary of the above problem.

**Theorem 1** Assume $A$, $B$ are squares, then we have

a) $\text{adj}(AB) = \text{adj} B \cdot \text{adj} A$; 

b) $\text{adj} XAX^{-1} = X(\text{adj} A)X^{-1}$; 

c) if $AB = BA$, then $(\text{adj} A)B = B(\text{adj} A)$.

14. (Continue) Use the conclusion from the above problem to prove that if a matrix $A$ (Symmetric or Hermitian) is positive definite, then $\text{adj} A$ is also a positive definite matrix. (Hint: positive definite is equivalent to all the eigenvalues are positive.)

15. Let $S$ be a symmetric invertible matrix of order $n$ all elements of which are positive. What is the largest possible number of zero elements of $S^{-1}$? Can you give an example for which this number is achieved?

8 AB and BA

16. Suppose $A$, $B$ are square matrices of the same dimension $n$.

a) If $A$ is invertible, show that $AB$ and $BA$ are always similar.

b) In general, are $AB$ and $BA$ always similar?

  Hint: If $n = 2$, this is actually true. You can show this easily. But be careful with higher dimensions.

c) Show that the characteristic polynomial of $AB$ is always equal to the characteristic polynomial of $BA$.

  Hint: You can first assume $A$ is invertible and use part a). The general result then follows from the deformation argument as in Problem 13. Again you don’t have to make this argument.

The point of this problem is you should see the contrast between part b) and part c) and understand that in general the equivalent relation of being similar is much more subtle than satisfying some equations. In particular, most invariants of matrices are algebraic functions of the entries of matrices which you can apply the deformation argument to. But deformation argument can not be applied to being similar.
9 Applications in Geometry

17. The ellipse of largest area that can be inscribed in an equilateral triangle is a circle.

a) What is the area of this circle if the side length of this triangle is \(l\)?

b) Now consider the right triangle whose vertices are at \((0,0)\), \((4,0)\), \((4,3)\). What is the largest area of an inscribed ellipse? Where is the center of the inscribed ellipse? What are the lengths of its semi major and semi minor axes?

Hint: Draw an equilateral triangle whose base is on the \(x\)-axis and extending from \((0,0)\) to \((0,4)\). Find the linear transformation sending \((4,0)\) to itself and sending the third vertex of the equilateral triangle which you just drew to \((4,3)\). What does the linear transformation you found do to areas? As a check on your work note that the area of an ellipse whose semi major and semi minor axes have lengths \(a\) and \(b\), respectively, is \(\pi ab\).