A model is a description of a system using mathematical concepts.

real world $\rightarrow$ model

model $\rightarrow$ solution

math
\[ \vec{v}_0 \]

\[ \vec{x}, \vec{v}, \vec{a}, \ t \]

\[ x'(t) = \vec{v} \quad \text{model} \]

\[ v'(t) = \vec{a} \quad \text{implicit} \]

\[ \vec{a} \cdot m = m \vec{g} \]

Three quantities 

independent variables \( t \)

dependent variables \( x, \vec{v}, \vec{a}, \)

parameters \( \vec{v}_0, m, \vec{g} \)
Solution

\[ \bar{X}(t) = \frac{1}{2} \bar{g} t^2 + \bar{u}_0 t + \bar{X}_0 \]

explicit

Population growth

Assumption

The rate of growth of population is proportional to the size of the population.
Implicitly we have time \( t \) as the independent variable and \( P(t) \) as the population. The proportion \( \frac{dP}{dt} \) is proportional to the population size \( P \), hence \( \frac{dP}{dt} = kP \), where \( k \) is a constant parameter.

\[
\frac{dP}{dt} = kP \\
\frac{dp}{P} = k \, dt \\
P = Ae^{kt}
\]
It doesn't determine \( P(t) \) completely

\[
P(t) = A e^{kt}
\]

Need to know \( A \)

Let \( t = 0 \)

\[
P(0) = A e^{k \cdot 0} = A
\]

\[
A = P(0) = P_0.
\]

call it initial value
Terminology

\[ \frac{dP}{dt} = kP \]

ordinary differential equation

\[ P(0) = P_0 \]

is a first order initial value problem.

\[ P = A e^{kt} \]  general solution

\[ P = P_0 e^{kt} \]  a particular solution.
The solution is an exact solution. The model is oversimplified.

\[
\text{exponential growth}
\]

\[
\text{real population}
\]

When the population is small, the resources abundant, the model is good.
When the population large, X.
In the long run there should be a limit.

Assume if $P > N$, no enough resources $P$ decreases

1) $P$ small $\frac{dP}{dt} = kP$

2) $P > N$ $\frac{dP}{dt} < 0$

\[ \frac{dP}{dt} = k(\cdot ?) \cdot P \]

? 1) $P$ small (? $\approx 1$

? 2) $P > N$ ? $< 0$
Try \(1 - \frac{p}{N}\)

You may have a lot of other options.

\[
1 - \left(\frac{p}{N}\right)^2
\]

\[
2 - 2\frac{p}{N}
\]

The simplest one

\[
1 - \frac{p}{N}
\]
\[ \frac{dP}{dt} = k \left(1 - \frac{P}{N}\right)P \]

This is NOT linear ODE.

\[ k \left(1 - \frac{P}{N}\right)P \]

Two zeros

\[ P = 0 \]
\[ P = N \]

\[ \frac{dP}{dt} = 0 \] \hspace{1cm} \text{equilibrium.}
Two populations

\[ F(t) \text{ fox} \]
\[ R(t) \text{ rabbits} \]

1) If no fox, the growth rate of \( R(t) \) is proportional to \( R(t) \)

2) Fox eat rabbits
   
   The rate rabbits are eaten is proportional to the product of \( F(t) \) and \( R(t) \).
3) Without rabbits

\[ F(t) \rightarrow \text{proportion to } F(t) \]

4) The rate foxes are born proportional to the rate rabbits are eaten

\( (\text{product of } R \text{ and } F) \)

1) \( \alpha \)

2) \( \beta \)

3) \( R \)

4) \( S \)
\[
\frac{dR}{dt} = \alpha R - \beta RF
\]
\[
\frac{dF}{dt} = -\gamma F + \delta RF
\]

Predator - Prey system
Focus on

\[ \frac{dy}{dt} = f(t, y) \]

\[ y(t_0) = y_0. \]

First order
initial value problem
1) \[ \frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t} \]

2) \[ \frac{dy}{dt} = t^3 + \sin t \]

3) \[ \frac{dP}{dt} = k \left( 1 - \frac{P}{N} \right) P \]

4) \[ \frac{dy}{dt} = y^2 \]

5) \[ \frac{dy}{dt} = \frac{\sin(y-t)}{\sin(y-t) + 8} \]
2) is the simplest

The right hand side

\( f(t, y) \) doesn't have \( y \)

\[ \frac{dy}{dt} = t^3 + \sin t \]

\[ y = \int t^3 + \sin t \, dt \]

\[ = \frac{1}{4} t^4 - \cos t + C \]

\[ f(t, y) = g(t) \] no \( y \)
\[ \frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t} \]

\[ \frac{dy}{y^2 - 1} = \frac{dt}{t^2 + 2t} \]

\[ \int \frac{dy}{y^2 - 1} = \int \frac{dt}{t^2 + 2t} \]

\[ \frac{1}{y^2 - 1} = \frac{1}{(y+1)(y-1)} = \frac{1}{2} \left( \frac{1}{y-1} - \frac{1}{y+1} \right) \]

\[ \frac{1}{t^2 + 2t} = \frac{1}{t(t+2)} = \frac{1}{2} \left( \frac{1}{t} - \frac{1}{t+2} \right) \]

\[ \int \left( \frac{1}{y-1} - \frac{1}{y+1} \right) dy = \int \frac{1}{t} - \frac{1}{t+2} \, dt \]
\[ \ln \left| \frac{y-1}{y+1} \right| = \ln \left| \frac{t}{t+2} \right| + C \]

\[ \left( \frac{y-1}{y+1} \right) = A \left( \frac{t}{t+2} \right) \]

\[ (y-1)(t+2) = A(y+1) t \]

\[ (t+2 - At)y = t+2 + At \]

\[ y = \frac{t+2 + At}{t+2 - At} \]

For example \( A = 1 \)

\[ y = t + t \]

\( A = 0 \quad y = 1 \)
\[
\frac{dP}{dt} = kP(1 - \frac{P}{N})
\]

\[
\frac{dP}{P(1 - \frac{P}{N})} = k\,dt
\]

\[
\int \left( \frac{1}{P} - \frac{1}{N - P} \right) d\frac{P}{N} = \int k\,dt
\]

\[
\ln \frac{P}{N} - \ln \left(1 - \frac{P}{N}\right) = kt + C
\]

Assume \( P < N \)

\[
\frac{\frac{P}{N}}{1 - \frac{P}{N}} = Ae^{kt} \quad \frac{P}{N} = \frac{Ae^{kt}}{1 + Ae^{kt}}
\]
\[
\frac{dy}{dt} = y^2
\]

\[
\int \frac{dy}{y^2} = \int dt
\]

\[
-\frac{1}{y} = t + c
\]

\[
y = -\frac{1}{t + c}
\]

What if \(y(0) = 0\)?

\[
y(0) = -\frac{1}{c} \neq 0 \text{ no solution?}
\]

But \(y(t) = 0\) is a solution!
\[
\frac{dy}{dt} = y^2
\]

\[
\Rightarrow \quad \frac{dy}{y^2} = dt
\]

Assumed \( y \neq 0 \)

But \( y=0 \) is an equilibrium.

Separable equations

\[
\int f(t,y) = g(t) h(y)
\]

\[
\frac{dy}{h(y)} = g(t) dt
\]
Of course we can also have

\[ f(t, y) = h(y) \rightarrow t \]

\[ \frac{dy}{h(y)} = \delta t \]
5) is not separable

\[ \frac{dy}{dt} = \frac{\sin(y-t)}{\sin(y-t) + 8} \]

\[ u = y - t \]

\[ \frac{du}{dt} = \frac{dy}{dt} - 1 \]

\[ \frac{du}{dt} + 1 = \frac{\sin u}{\sin u + 8} \]

\[ \frac{du}{dt} = -\frac{8}{\sin u + 8} \]

\[ (\sin u + 8) \, du = -8 \, dt \]

separable