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1.3 slope field
4) \[ \frac{dy}{dt} = y^2 \]

5) \[ \frac{dy}{dt} = \frac{\sin(y-t)}{\sin(y-t) + 8} \]

4) is separable

\[ \frac{dy}{y^2} = dt \]

\[ \int \frac{dy}{y^2} = \int dt \]

\[ -\frac{1}{y} = t + c \]

\[ y = -\frac{1}{t + c} \]
What is the solution of the initial value problem

\[ \frac{dy}{y^2} = dt \]

\[ y(0) = a \ ? \]

If \[ y = -\frac{1}{t+c} \]

\[ y(0) = -\frac{1}{c} \neq 0. \]

However \[ y(t) = 0 \]

is a solution
When we do

\[ \frac{dy}{dt} = y^2 \implies \frac{dy}{y^2} = dt \]

We are making an assumption

\[ y \neq 0 \]

In fact, \( y = 0 \) is an equilibrium.
5) \[
\frac{dy}{dt} = \frac{\sin(y-t)}{\sin(y-t)+8}
\]

not separable

\[u = y-t\]

\[\frac{du}{dt} = \frac{dy}{dt} - 1\]

\[\frac{du}{dt} + 1 = \frac{\sin u}{\sin u + 8}\]

\[\frac{du}{dt} = -\frac{8}{\sin u + 8}\]

\[\sin u + 8 \ du = -8 \ dt\]

\[-\cos u + 8 u = -8t + C\]
\[ y = u + t \]

\[- \cos (y - t) + 8y = C \]

This is an implicit form of \( y(t) \)
What if $f(t,y)$ is more complicated?

We don't have a general way of solving

$$\frac{dy}{dt} = f(t,y)$$
Geometric meaning

If we have a solution $y(t)$ passing $(t_1, y_1)$ in the $(t,y)$ plane.

The slope of the tangent line at $(t_1, y_1)$ is $f(t_1, y_1)$
Given \( f(t,y) \)

we may not know the solution,

we know the slope of the tangent line of the solution.
\[ \frac{dy}{dt} = y - t \]

\[ f(t, y) = y - t \]
Wolfram Alpha

Solve the equation

\[ \frac{dy}{dt} = y - t \]

\[ u = y - t \]

\[ \frac{du}{dt} = \frac{dy}{dt} - 1 = u - 1 \]

\[ \ln |u - 1| = t + C \]
\[ u - 1 = Ae^t \]
\[ y = t + 1 + Ae^t \]

Two special cases

\[ f(t, y) = g(t) \]
\[ f(t, y) = h(y) \]
\[ \frac{dy}{dt} = g(t) \]

The slope field is \( g(t) \) at \((t,y)\) doesn't depend on \( y \).

All the tangent lines on the vertical line are parallel.
\[
\frac{dy}{dt} = \frac{1}{t}
\]

\[y = ln|t| + C\]

moving along the vertical line is changing C
2) \( \frac{dy}{dt} = h(y) \) autonomous

slope at \((t, y)\) is

\(h(y)\) doesn't depend on \(t\)

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tangent lines along each horizontal line
\frac{dy}{dt} = y^2 - 1
\frac{dy}{y^2 - 1} = dt
\frac{1}{2} \left( \frac{1}{y-1} - \frac{1}{y+1} \right) dy = dt
\frac{1}{2} \left( \ln |y-1| - \ln |y+1| \right) = t + C
\frac{y-1}{y+1} = Ae^{2t}
\cdot y = \frac{1 + Ae^{2t}}{1 - Ae^{2t}}

\text{translate along } t
\text{still a solution varying } A
Definition

\[ \frac{dy}{dt} = f(t, y) \]

is called **autonomous**

if \( f(t, y) = h(y) \)

doesn't depend on \( t \)

If \( y(t) \) is a solution

of an autonomous equation

then \( y(t+c) \) is also a solution
Example

\[
\frac{dy}{dt} = e^{\frac{y^2}{2}} \sin^2 y
\]

This is autonomous, thus separable

\[
\frac{dy}{e^{\frac{y^2}{2}} \sin^2 y} = dt
\]

\[
\int \frac{dy}{e^{\frac{y^2}{2}} \sin^2 y} = t
\]

???
Note that

\[ y = n\pi \quad n = \ldots -1, 0, 1, \ldots \]

\[ \sin^2 y = 0 \]

\[ y = n\pi \] are equilibrium

We don't know what's happening here.

But the solutions must be between the lines
Moreover between lines

\[ y = (n-1) \pi \]

and \( y = n\pi \)

the slope

\[ e^{\frac{y^2}{2}} \sin^2 y \]

are positive

\[ \text{guess} \]