Solving equations

1st order\[ \frac{dy}{dt} = a(t)y + b(t) \]

\[ \frac{dy}{dt} = 2y + e^t \]

1st order constant coefficient\[ \frac{dy}{dt} - 2y = e^t \]

\[ p(\lambda) = \lambda - 2 \]

\[ \lambda = 2 \]

\[ y(t) = Ae^t \]

\[ Ae^t - 2Ae^t = e^t \]

\[ A = -1 \]
\[ \frac{dy}{dt} - 2y = e^{2t} \]

\[ p(\lambda) = \lambda - 2 \]

Assume
\[ y(t) = Ae^{2t} \]

\[ 2Ate^{2t} + Ae^{2t} - 2Ate^{2t} = e^{2t} \]

\[ A = 1 \]

\[ y(t) = t e^{2t} \]
Integrating factor

\[ t^2 y' - 4 \frac{t}{y} = t^7 \sin t \]

\[ y' = 4 \frac{t}{y} + t^5 \sin t \]

\[ I(t) = \left[ a(t) \right] = e^{- \int a(t) \, dt} = e^{- \int \frac{4}{t} \, dt} = e^{-4 \ln t - t - 4} \]

\[ t^{-4} y' - \frac{4}{t^3} y = t \sin t \]

\[ (y t^{-4})' = t \sin t \]
\[ t^{-4} y = \int t \sin(t) \, dt \]
\[ = \int t \, d(-\cos t) \]
\[ = -t \cos t + \int \cos t \, dt \]
\[ = \sin t - t \cos t \]
\[ y = t^{-4} (\sin t - t \cos t) \]

See Notes
\[ q/12 - 9/14 \]
Separable

\[
\frac{dy}{dt} = p(y) q(t)
\]

\[
\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 1}
\]

\[
\frac{dy}{y^2 - 1} = \frac{dt}{t^2 + 1}
\]

\[
\int \frac{dy}{y^2 - 1} = \int \frac{dy}{(y-1)(y+1)}
\]

\[
= \int \frac{1}{2}(\frac{1}{y-1} - \frac{1}{y+1}) dy
\]

\[
= \frac{1}{2} \left[ \ln(y-1) - \ln(y+1) \right]
\]
\[ \int \frac{dt}{t^2 + 1} = \tan^{-1} t + C \]

\[ \frac{1}{2} \left( \ln (y - 1) - \ln (y + 1) \right) = \tan^{-1} t + C \]

\[ \ln \frac{y - 1}{y + 1} = 2\tan^{-1} t + C' \]

\[ \frac{y - 1}{y + 1} = A e^{2\tan^{-1} t} \]

\[ y - 1 = A e^{2\tan^{-1} t} y + A e^{2\tan^{-1} t} \]

\[ (1 - Ae^{2\tan^{-1} t}) y = 1 + Ae^{2\tan^{-1} t} \]

\[ y = \frac{1 + Ae^{2\tan^{-1} t}}{1 - Ae^{2\tan^{-1} t}} \]
See 8129 Notes
2×2 linear system

\[
\frac{d\vec{Y}}{dt} = A\vec{Y}
\]

1) Two distinct real roots
2) Complex roots
3) Repeated roots

The type of equilibrium in each case
\[ A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \]

1) Eigenvalue

\[ p(\lambda) = \lambda^2 - 8\lambda + 25 \]

\[ \lambda = \frac{8 \pm \sqrt{-36}}{2} \]

\[ = 4 \pm 3i \]

\[ \alpha = 4 > 0 \]

spiral source
2) **Eigenvalue**

\[ \lambda = 4 + 3i \]

\[ A - \lambda I = \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \]

\[ \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \vec{u} + i\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ \vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ e^{4t} \left( \cos 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \]

\[ e^{4t} \left( \cos 3t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \]
\[ A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \]

1) Eigenvalue

\[ p(\lambda) = \lambda^2 - 8\lambda + 7 \]
\[ = (\lambda - 1)(\lambda - 7) \]
\[ \lambda_1 = 1 \quad \lambda_2 = 7 \quad \text{distinct real roots} \]
\[ > 0 \quad > 0 \quad \text{source} \]

2) Eigenvalue

\[ \lambda = 1 \quad A - \lambda I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \]
\[ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
\[ \lambda = 7 \quad A - \lambda I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \]
\[ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
\[ e^t [-1], e^{7t} [1] \]

phase portrait

1) You need to show the special directions.

2) General solution right asymptotic behavior.
\[ A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \]

1) Eigenvalue.

\[ p(\lambda) = \lambda^2 - 6\lambda + 9 \]

\[ = (\lambda - 3)^2 \]

Repeated roots.

2) Generalized eigenvector

\[ \vec{v}_0 = \begin{bmatrix} a \\ b \end{bmatrix}, \text{ arbitrary.} \]

\[ \vec{v}_1 = (A - \lambda I) \vec{v}_0 \]

\[ = \begin{bmatrix} -1 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a + b \\ -a + 4b \end{bmatrix} \]
\[ e^{3t} \vec{v}_0 + t e^{3t} \vec{v}_i \]

\[ = e^{3t} \begin{bmatrix} a \\ b \end{bmatrix} + t e^{3t} \begin{bmatrix} -a+b \\ -a+b \end{bmatrix} \]

phase portrait

1) Special direction \( \vec{v}_i \)

- \((-1, 0)\)
- \((-2, 1)\)

look at \((1, 0)\), \((2, -1)\)
\[
A = \begin{bmatrix}
1 & 3 \\
2 & 6
\end{bmatrix}
\]

1) Eigenvalue

\[
\Phi(\lambda) = \lambda^2 - 7\lambda = \lambda(\lambda - 7)
\]

\[
\lambda_1 = 0, \quad \lambda_2 = 7
\]

2) Eigenvector

\[
\lambda_1 = 0 \quad \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \rightarrow \text{a line of equilibria}
\]

\[
\lambda_2 = 7 \quad \begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]
phase portrait

(3, -1)
equilibrria

Notes 10/10 - 10/12
Qualitative Study.

Existence + Uniqueness Thm.

9/5 9/26

Autonomous system

phase line

phase plane

Bifurcation
\[ \frac{dy}{dt} = 2y(1-y) \]

phase line
Determine the bifurcation

$$\frac{dy}{dt} = y^6 - 2y^3 + \alpha$$

Necessary condition

$$\frac{dy}{dt} = f_\mu(y)$$

1) $f_\mu(y_0) = 0$ $y_0$ is an equilibrium

2) $f_\mu'(y_0) = 0$

$$\begin{cases} y^6 - 2y^3 + \alpha = 0 \\ 6y^5 - 6y^2 = 0 \end{cases}$$

$\alpha = 0$ $y_0 = 0$ $y_0 = 1$ $\text{or } \alpha = 1$
\[ \alpha = 0, \quad \alpha = 1 \]
possible bifurcation values
still need to check

\[ y^6 - 2y^3 = y^3(y^3 - 2) \]

\[ \alpha = 0 \text{ is not a bifurcation} \]
\[ \alpha = 1 \text{ is a bifurcation} \]

\[ \alpha = 1 \quad y^6 - 2y^3 + 1 = (y^3 - 1)^2 \]
Bifurcation diagram.

9/7 - 9/12
Example

\[ \frac{dy}{dt} = y^2 (1-y) - \mu \]

Bifurcation

\[
\begin{cases}
    f_\mu (y) = 0 \\
    f_\mu' (y) = 0 \\
    y^2 (1-y) - \mu = 0 \\
    2y (1-y) + y^2 (-1) = 0
\end{cases}
\]

\[ 2y - 3y^2 = 0 \]

\[ y = 0 \text{ or } y = \frac{2}{3} \]

\[ \mu = 0 \text{ or } \mu = \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{27} \]
Both are equilibria.
phase plane

$2 \times 2$ linear system

See before

Predator–Prey system

See below.
Famous system you need to know

Logistic model

\[ \frac{dP}{dt} = k \left( 1 - \frac{P}{N} \right) P \]

or \[ \frac{dy}{dt} = k \left( 1 - \frac{y}{M} \right) y \]

meaning of the model

phase line
Solution graph

\[ N \quad t \]
Predator - Prey

\[
\frac{dR}{dt} = \alpha R - \beta RF
\]

\[
\frac{dF}{dt} = -rF + sRF
\]

\( \alpha, \beta, r, s > 0 \)

meaning of the model

\[
\begin{align*}
R & \quad \uparrow \\
\downarrow & \quad F
\end{align*}
\]