1.6 Equilibria and phase line

1.7 Bifurcation
Autonomous equation

\[ \frac{dy}{dt} = f(y) \]

We learnt how to use the slope field to sketch the solutions.

Since autonomous, the slope field

// // //

parallel if y coordinates the same.
have a simpler tool to study the behavior

phase line

a vertical line contains all the information
Example logistic model

\[ \frac{dy}{dt} = (1 - y) y \]

Equilibria

\[(1 - y)y = 0\]

\[y = 0 \text{ or } 1\]

\[f(y) < 0 \quad y = 1\]

\[f(y) > 0 \quad y = 0\]

\[f(y) < 0\]
By existence and uniqueness theorem

\[ y = 1 \]
\[ f(y) > 0 \]
\[ y = 0 \]

Phase line
\( f(y) \) \( f'(y) \) continuous for all \( y \) \( \text{(continuously differentiable)} \)

so that the uniqueness theorem holds.

\[ \uparrow \] \( y \) line

1) Find roots \( f(y) = 0 \) equilibria

2) Find intervals where \( f'(y) > 0 \) solutions increasing \( \uparrow \)
3) Find intervals where \( f(y) < 0 \) and solutions decreasing.

\[
\frac{dy}{dt} = \frac{(y-2)(y+3)}{}
\]
\[
\frac{dy}{dt} = y \cos y
\]

\(3 \frac{\pi}{2}\)

\(\frac{\pi}{2}\)

\(0\)

\(-\frac{\pi}{2}\)

\(-\frac{3\pi}{2}\)

1) \(y \cos y = 0\)

\(y = 0 \text{ or } y = \frac{\pi}{2} + k\pi\)
Phase lines are enough to tell us about the limiting behavior of solutions.

\[ \frac{dW}{dt} = (2 - w) \sin w \]

\( (2 - w) \sin w = 0 \) \quad \text{or} \quad w = 2 \pi k \quad (k \text{ is an integer})
Example

\[
\frac{dy}{dt} = (y-1)(y-2)^2(y-3)^3
\]

Sketch the solution for \(y(0) = 2.5\)
\[
\frac{dy}{dt} = f(y)
\]

continuously differentiable

for all \( y \)

1) \( f(y(0)) = 0 \)
   \( y = y(0) \) equilibrium

2) \( f(y(0)) > 0 \)
   \( y \) increasing either \( y(t) \to \infty \)
   or \( y(t) \to \) the equilibrium above it.

3) \( f(y(0)) < 0 \)
   \( y \) decreasing either \( y(t) \to -\infty \)
   or the equilibrium below.
Example

\( y(t) \) may go to \( +\infty \)

\[
\frac{dy}{dt} = (1+y)^2
\]
Example

\[ f(y) \text{ not continuously differentiable at some point (Be careful)} \]

\[ \frac{dy}{dt} = \frac{1}{1-y} \]

\[ y \text{ not continuous at 1} \]
Suppose \( f(y) \) is continuously differentiable.

Classification of equilibria

\[ \leftrightarrow \text{classification of zeros of } f(y) \]
sink

Source

node
Linearization Theorem

Type of zeros are classified by $f'(y_0)$

- $f'(y_0) < 0$  sink
- $f'(y_0) > 0$  source
- $f'(y_0) = 0$  need more information
  undetermined
A node \( y_0 \) must have 
\[ f'(y_0) = 0 \]

But \( f'(y_0) = 0 \) doesn't imply \( y_0 \) node

Example

\[ f(y) = y^3 \quad y_0 = 0 \]
\[ f'(y_0) = 0 \quad \text{Source} \]
Example

\[ \frac{dy}{dt} = y \left( \cos (y^5 + 2y) - 27\pi y^4 \right) \]

\[ = h(y) \]

\[ h(0) = 0 \quad y_0 = 0 \]

sink? source? node?

\[ h'(y) = \cos (y^5 + 2y) - 27\pi y^4 \]

\[ + y \left( -\sin (y^5 + 2y) \cdot \left( \ldots \right)^1 - 27\pi \cdot 4y^3 \right) \]

\[ = 0 \quad \text{at } y. \]

\[ h'(0) = 1 - 0 = 1 > 0 \]

source.
Recall dependent variable

independent variable

parameters

Let parameters vary

→ a family of ODE's.

Does the qualitative behavior change a lot?

Usually not

But occasionally small change

→ drastic change—bifurcation!
\[
\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)
\]

As \(N \to \infty\), the population size becomes very large, and the term \(1 - \frac{P}{N}\) approaches 1. Thus, the growth rate \(\frac{dP}{dt}\) approaches \(kP\), indicating that the population grows exponentially. Conversely, as \(N \to 0\), the term \(1 - \frac{P}{N}\) approaches 0, and the growth rate \(\frac{dP}{dt}\) approaches 0, indicating that the population growth slows down or stops due to limited resources.
One parameter varies

→ one-parameter family

\[
\frac{dy}{dt} = y^2 - 2y + \mu
\]

\( \mu \) is the parameter

\[
f_\mu(y) = y^2 - 2y + \mu
\]

\( \mu \) varies

Type of zeros varies
\[ y^2 - 2y + m = 0 \]

\[ 4 - 4m \]

\[ \begin{align*}
\text{> 0} & \quad \text{two distinct real solutions} \\
\text{= 0} & \quad 1 \ \text{solution} \\
\text{< 0} & \quad \text{no real solution}
\end{align*} \]
$\mu = -3.$

$$y^2 - 2y - 3 = 0$$

$$(y - 3)(y + 1) = 0$$
\[ \mu = 1 \]
\[ y^2 - 2y + 1 = 0 \]
\[ (y - 1)^2 = 0 \]
\[ m = 2 \]

\[ y^2 - 2y + 2 = (y - 1)^2 + 1 > 0 \]

no equilibria
\( \mu < 1 \)

bifurcation

\( \mu = 1 \)

\( \mu > 1 \)
We say $m = 1$ is a **bifurcation value**

See it from the phase line

$\begin{array}{c} 
m < 1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
m = 1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
m > 1 \\
\end{array}$
The Bifurcation diagram
μ - y plane
for fixed μ (draw the phase line for f(μ,y))
\[ \frac{dy}{dt} = g_\alpha(y) = y^3 - \alpha y \]

\[ = y(y^2 - \alpha) \]

If \( \alpha < 0 \) \( y^2 - \alpha > 0 \).

\[ g'_\alpha(y) = 3y^2 - \alpha > 0 \]

\( y = 0 \) is the source
$\alpha = 0$ still source

$\alpha > 0$

$\alpha = 0$ bifurcation value
How to find the bifurcation values?

If \( f_{\mu_0}(y_0) = 0 \) equilibrium

and \( f'_{\mu_0}(y_0) > 0 \)

or \( f'_{\mu_0}(y_0) < 0 \)

assume \( f' \) continuous for \( \mu \)

vary \( \mu_0 \rightarrow \mu \)

If \( \mu \) is close to \( \mu_0 \)
If \( \mu_0 \) is a bifurcation value, the necessary condition is there is \( y_0 \) such that

\[
\begin{align*}
 f_{\mu_0}(y_0) &= 0 \\
 f'_{\mu_0}(y_0) &= 0
\end{align*}
\]
Example

\[ \frac{dy}{dt} = y(1-y)^2 + \mu \]

\[ f_\mu(y) = y(1-y)^2 + \mu \]

\[ f_\mu^{-1}(y) = (1-y)^2 + 2y(1-y)(1-\mu) \]

\[ = (1-3y)(1-y) \]

\[ f_\mu'(y) = 0 \Rightarrow y=1 \text{ or } y=\frac{1}{3} \]

Need \( \mu \) such that

1 or 3 is an equilibrium.
\[ \int \mu (1) = \mu = 0 \]

\[ \mu = 0 \]

\[ \int \mu \left( \frac{1}{3} \right) = \frac{1}{3} \left( \frac{2}{3} \right)^2 + \mu = 0 \]

\[ \mu = -\frac{4}{27} \]

Check if \( \mu = 0, -\frac{4}{27} \) bifurcation.
Near 0

\[ y(1-y)^2 \]

\[ m > 0 \]

\[ m < 0 \]

Near \(-\frac{4}{27}\)

1 2 3

1 2 3
Both are bifurcation values.