Midterm Exam #1 for Math 420, Fall 2016

Name (printed): ____________________________

My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this midterm examination.

Signature: ____________________________ Date: ____________________________

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- You have 50 minutes for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate. Answers with little or no justification will get no credit.
- You may use both sides of one 8.5 by 11 inch sheet of notes.
- **NO** books, laptops, cell phones, calculators, or any other electronic devices may be used during the exam.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.
- Box your final answers.
1. (10 points) Find the general solution to the following differential equation:

\[
\frac{dy}{dt} = 2y + e^t.
\]
2. Consider the following linear system of differential equations:

\[ \frac{d\vec{Y}(t)}{dt} = A\vec{Y}(t), \quad \text{where } \vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}. \quad (1) \]

(a) (2 points) Write down the characteristic polynomial of the matrix \( A \). Find the eigenvalues of \( A \).

(b) (4 points) Find the general solution to the differential equation given in equation (1) above.

(c) (2 points) For the differential equation given in equation (1), consider the initial condition \( \vec{Y}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \). Find the solution to this initial value problem.

(d) (2 points) Find the equilibrium solutions to the differential equation given in equation (1) above. For each equilibrium point, classify their types (i.e., whether they are a sink or a source or a spiral sink etc).
3. Consider the following linear system of differential equations:

\[
\frac{d\vec{Y}(t)}{dt} = A\vec{Y}(t), \quad \text{where } \vec{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}.
\] (2)

(a) (5 points) Find the general solution to the differential equation given in equation (2) above.

(b) (5 points) Draw the phase portrait for the differential equation given in equation (2) above.
4. Consider the following one parameter family of ordinary differential equations:

\[ \frac{dy}{dt} = y^2(1 - y) - \mu. \]

(a) (6 points) Locate all the bifurcation points of this family of ODEs. 
(Note that if you are using the bifurcation test, your solution must clearly explain why the points obtained via the bifurcation test are bifurcation points.)

(b) (4 points) Draw the bifurcation diagram for the above one parameter family of ODEs.
5. Suppose \( \vec{\phi}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \) satisfies the following differential equation:

\[
\frac{d\vec{\phi}(t)}{dt} = B\vec{\phi}(t),
\]

where \( B \) is a \( 2 \times 2 \) matrix with real entries. Also suppose that the following condition holds:

\[
\lim_{t \to -\infty} x(t) = \infty, \quad \lim_{t \to \infty} x(t) = \infty.
\]

Let \( \lambda_1 \) and \( \lambda_2 \) denote the eigenvalues of \( B \).

(a) (4 points) Only one of the following options is correct. Chose the correct option.

i. The eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are both real with \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \).

ii. The eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are both real with \( \lambda_1 < 0 \) and \( \lambda_2 > 0 \).

iii. The eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are both real with \( \lambda_1 = 0 \) and \( \lambda_2 > 0 \).

iv. The eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are both complex.

(b) (3 points) Let \( \text{Det}(B) \) denote the determinant of the matrix \( B \). Only one of the following options is correct. Chose the correct option.

i. \( \text{Det}(B) = 0 \).

ii. \( \text{Det}(B) > 0 \).

iii. \( \text{Det}(B) < 0 \).

iv. \( \text{Det}(B) \) is a complex number.

(c) (3 points) Only one of the following options is correct. Chose the correct option.

i. The origin is a sink.

ii. The origin is a source.

iii. The origin is a saddle.

iv. The origin is a spiral source.